Optimum Linear Transmitter Design for MIMO Systems with Two QPSK Data Streams

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***Abstract*— We present the optimum design of a linear transmitter for a multi-input multi-output communication system when the input data consists of two QPSK streams and the receiver performs maximum likelihood detection. The transmitter design is optimal in the sense that it minimizes the worst-case pairwise error probability. We prove that the resulting linear transmitter constructs, as a function of the two input QPSK streams, a new signal constellation. Moreover, we show that this new signal constellation is such that the minimum distance among the points of the received constellation is maximized and, indirectly, the number of neighbors at minimum distance is also maximized.**

I. Introduction

Multi-input multi-output (MIMO) communications can achieve very important improvements in system performance due to the diversity provided by the multiple parallel channels that can be established among the transmit and receive antennas. In this scenario, much research effort has been put on the design of appropriate transmitters and receivers for such communication systems, trying to exploit and get all the inherent advantages and potentials of the MIMO channel.

The design of the transmitter depends on two important points. The first one is the quantity and the quality of the channel state information (CSI) available at the transmitter (usually, the CSI is assumed to be known at the receiver). On the other hand, the design also depends on the kind of receiver employed and, consequently, on the figure of merit adopted to measure the system performance.

Several papers have been devoted to the case of having no CSI at the transmitter, leading to the so called space-time codes (STC), which can be block [1], [2] or convolutional [3]. In [3], some convolutional codes are proposed using the rank and the determinant criteria. These criteria are indirectly related to the worst pairwise error probability (PEP), which is usually taken as the performance measure when a maximum likelihood (ML) detector is employed at the receiver.

The case of assuming a perfect CSI at the transmitter has been extensively studied by several authors. The differences among the works rely on the kind of transmitter and receiver, which may be linear or nonlinear, and the performance measure. The case of linear transmitters and receivers has been solved for most of the already known performance measures in [4], and has been further generalized, under the framework of convex optimization, in [5]. It has also been shown that the obtained performances can be improved when nonlinear stages are permitted or when the complexity of the receiver is increased, taking as the objective the minimization of the bit error rate (BER) (see *e.g.* the application of the nonlinear receiver MMSE-VBLAST in [6]).

Although the case of having a perfect CSI at both sides of the system has been studied deeply, to the best of our knowledge the only work in which the transmitter is designed assuming that a perfect ML receiver is employed is [7]. The utilization of a ML receiver is motivated by recent efficient implementations such as the optimal sphere decoder [8], [9], or the schemes based on semidefinite relaxation (SDR) [10], [11], [12], which reduce the computational complexity by sacrificing optimality.

In this paper, we address the problem of designing a linear transmitter for a MIMO system with perfect CSI at both sides, when a ML detector is used at the receiver. According to this, a meaningful objective is the minimization of the worst PEP for any possible pair of transmitted symbol vectors. The problem is mathematically formulated as a maximin optimization problem, which in general is quite complicated. Besides, in this particular case, an additional complexity relies on the fact that one of the optimization variables in the problem is discrete. Because of this, the optimum design is found for the concrete case of transmitting two QPSK symbol sequences. The resulting linear transmitter is responsible for constructing a new signal constellation (based on the two QPSK symbols) which is sent through one or two of the MIMO channel eigenmodes depending on the relation between the corresponding eigenvalues. We also show that, under the criterion of minimizing the maximum PEP, the new signal constellation is such that the minimum distance among the points of the received constellation is maximized and, indirectly, the number of neighbors at minimum distance is maximized.

In Section II we give an overview on the MIMO communication system considered in this work. Next, in Section III we review the ML detector and give an expression for the PEP. The problem statement as the minimization of the worst-case PEP and its solution are given in Section IV. In Section V some simulations results, which show the performance of the proposed scheme, are presented. Finally, some conclusions and further remarks are drawn in Section VI.

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II. System Model

A narrowband multiplexing system with n_T transmit and n_R receive antennas corrupted with additive Gaussian noise is considered. Let us define $\mathbf{x} \in \mathbb{C}^{n_T}$ as the transmitted signal, where $[\mathbf{x}]_k$ represents the transmitted signal through k-th antenna. We also define $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ as the channel matrix, where $[\mathbf{H}]_{jk}$ represents the baseband equivalent path gain from the k-th transmitter to the j-th receiver. Finally, $\mathbf{w} \in \mathbb{C}^{n_R}$ is defined as the noise vector, where $[\mathbf{w}]_j$ represents the noise component received at the j-th antenna. The noise vector is modeled as a spatially white circularly symmetric Gaussian distributed random vector, with $\mathbb{E}|[\mathbf{w}]_j|^2 = \sigma^2, \forall j$. The received signals vector, $\mathbf{y} \in \mathbb{C}^{n_R}$, for this model can be expressed as $\mathbf{y} = \mathbf{Hx} + \mathbf{w}$.

As it is stated in the introduction, the transmitted signal is obtained from a linear combination of two QPSK data symbols, which are stacked to form the data symbols vector $\mathbf{s} \in \mathcal{M}^2$, where $\mathcal{M}$ represents the QPSK constellation, *i.e.*,

$$\mathcal{M} \equiv \left\{ \frac{1+\mathrm{i}}{\sqrt{2}}, \frac{1-\mathrm{i}}{\sqrt{2}}, \frac{-1+\mathrm{i}}{\sqrt{2}}, \frac{-1-\mathrm{i}}{\sqrt{2}} \right\}.$$

Consequently, the transmitted signal is given by $\mathbf{x} = \mathbf{Bs}$, where $\mathbf{B} \in \mathbb{C}^{n_T \times 2}$ represents a generic linear transformation. Throughout this work we assume that the receiver employs a maximum likelihood (ML) detector. See Fig. 1 for a graphical representation of the described communication scheme.

III. Maximum Likelihood Receiver

Since the receiver performs ML detection, the estimate of the data symbols vector is given by the vector $\hat{\mathbf{s}}$ maximizing the log-likelihood function, $\mathcal{L}(\mathbf{s}) = -\|\mathbf{y} - \mathbf{HBs}\|^2$, *i.e.*,

$$\hat{\mathbf{s}} = \arg\max_{\mathbf{s}} \mathcal{L}(\mathbf{s}).$$

In our case, there are $|\mathcal{M}^2| = 16$ different data symbols vectors, which can then be indexed from 1 to 16 as $\mathbf{s}_n$, where $n \in \{1, 2, \ldots, 16\}$. The probability of deciding in favor of $\mathbf{s}_m$ when $\mathbf{s}_n$ is actually transmitted is denoted by $\mathsf{pep}_{n,m}$, and it is given by the probability that $\mathcal{L}(\mathbf{s}_m) > \mathcal{L}(\mathbf{s}_n)$ conditioned on the fact that $\mathbf{s}_n$ is transmitted. From, *e.g.*, [13, p.265] we recall the well-known expression for this probability

$$\mathsf{pep}_{n,m} = \Pr\left(\mathcal{L}(\mathbf{s}_m) > \mathcal{L}(\mathbf{s}_n) | \mathbf{s}_n\right) = \mathcal{Q}\left(\sqrt{\frac{d_{n,m}^2}{2\sigma^2}}\right), \quad (1)$$

where $\mathcal{Q}(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-t^2/2) \mathrm{d}t$ and $d_{n,m}^2$ represents the squared distance between the received constellation points $\mathbf{HBs}_n$ and $\mathbf{HBs}_m$, *i.e.*,

$$d_{n,m}^2 = \|\mathbf{HBs}_n - \mathbf{HBs}_m\|^2 = \mathbf{e}_{n,m}^H \mathbf{B}^H \mathbf{R_H} \mathbf{B} \mathbf{e}_{n,m}, \quad (2)$$

where $\mathbf{R_H} = \mathbf{H}^H\mathbf{H}$ and $\mathbf{e}_{n,m} = \mathbf{s}_n - \mathbf{s}_m$.

Now, we denote by $\mathcal{E}$ the set of all possible error vectors, $\mathcal{E} = \{\mathbf{e}_{n,m}\}$. From the definition of the error vector, $\mathbf{e}_{n,m} = \mathbf{s}_n - \mathbf{s}_m$, one easily sees that each component of $\mathbf{e}_{n,m}$ must belong to the set $\mathcal{S} \equiv \left\{0, \pm\sqrt{2}, \pm\mathrm{i}\sqrt{2}, \pm\sqrt{2} \pm \mathrm{i}\sqrt{2}\right\}$. Considering that the zero vector can not belong to $\mathcal{E}$, we obtain

$$\mathcal{E} = \mathcal{S}^2 \backslash \{\mathbf{0}\}. \quad (3)$$

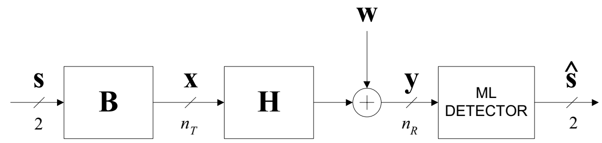

Fig. 1. MIMO communications scheme considered in this work.

The cardinal of $\mathcal{E}$ is $|\mathcal{E}| = |\mathcal{S}|^2 - 1 = 80$. This implies that the different error vectors can be indexed from 1 to 80, as $\mathbf{e}_q$, with $q = \{1, 2, \ldots, 80\}$. Since the expression for $d_{n,m}^2$ in (2) depends on the particular error vector, we can also index the set $\{d_{n,m}^2\}$, with the same label that we use for the error vectors, q. In the following, d_q^2 is utilized instead of $d_{n,m}^2$, and, equivalently, pep_q replaces $\mathsf{pep}_{n,m}$.

IV. Optimum Transmitter Design

We now present the design of the optimum linear transmitter, $\mathbf{B}^\star$. The criterion for optimality is to minimize the maximum (or worst-case) PEP with respect to the set of error vectors, or, formally

$$\mathbf{B}^\star = \arg\min_{\mathbf{B}} \max_{q} \mathsf{pep}_q, \quad (4)$$

where $q \in \{1, 2, \ldots, |\mathcal{E}|\}$. Moreover, the search space of the minimization with respect to $\mathbf{B}$ is restricted to matrices such that $\mathrm{Tr}\, \mathbf{BB}^H \leq P_T$, where P_T represents the maximum mean transmitted power. Notice that, due to the dependence given in (1), minimizing the worst-case PEP is equivalent to maximizing the squared minimum distance between the received constellation points:

$$\arg\min_{\mathbf{B}} \max_{q} \mathsf{pep}_q = \arg\max_{\mathbf{B}} \min_{q} d_q^2 = \arg\max_{\mathbf{B}} d_{\min}^2.$$

If we write explicitly the squared distance as a function of the transmission matrix $\mathbf{B}$ and the error vector $\mathbf{e}_q$ as in (2), the optimization problem in (4) is equivalently reformulated as

$$\mathbf{B}^\star = \arg\max_{\mathbf{B}} \min_{\mathbf{e}} \mathbf{e}^H \mathbf{B}^H \mathbf{R_H} \mathbf{B} \mathbf{e},$$

where $\mathbf{e} \in \mathcal{E}$ and $\mathrm{Tr}\mathbf{BB}^H \leq P_T$. Next, we solve this problem.

Proposition 1: Consider the following constrained maximin optimization problem

$$\begin{aligned} \max_{\mathbf{B}} \min_{\mathbf{e}} \quad & \mathbf{e}^H \mathbf{B}^H \mathbf{R_H} \mathbf{B} \mathbf{e}, \\ \text{s.t.} \quad & \mathbf{e} \in \mathcal{E}, \\ & \mathbf{B} \in \mathbb{C}^{n_T \times 2}, \ \mathrm{Tr}\, \mathbf{BB}^H \leq P_T, \end{aligned} \quad (5)$$

where $\mathbf{R_H} \in \mathbb{C}^{n_T \times n_T}$ is a positive semidefinite hermitian matrix with λ_1 and λ_2 being its two largest eigenvalues, with $\lambda_1 \geq \lambda_2$, and $\mathcal{E}$ is the same as defined in (3). It then follows that there is an optimal solution, $\mathbf{B}^\star$, which is given by $\mathbf{B}^\star = \mathbf{U_H} \mathbf{\Sigma} \mathbf{V}^H$, where $\mathbf{U_H} \in \mathbb{C}^{n_T \times 2}$ has as columns the eigenvectors of $\mathbf{R_H}$ corresponding to λ_1 and λ_2. In addition, defining

$$\lambda_c \equiv \frac{(\sqrt{3}-1)(3-2\sqrt{2})}{1+3\sqrt{3}-2\sqrt{6}} \approx 9.683 \cdot 10^{-2},$$

the optimal solution is completed with

- If $\lambda_2/\lambda_1 < \lambda_c$, then

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{P_T} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{V}^H = \begin{bmatrix} \cos\theta_T & -\mathrm{e}^{\mathrm{i}\phi_T}\sin\theta_T \\ \mathrm{e}^{-\mathrm{i}\phi_T}\sin\theta_T & \cos\theta_T \end{bmatrix},$$

where $\phi_T = \arccos\left(\frac{3+\sqrt{3}}{2\sqrt{6}}\right)$ and $\theta_T = \arctan\left(\frac{\sqrt{6}}{3+\sqrt{3}}\right)$.

- If $\lambda_2/\lambda_1 \geq \lambda_c$, then

$$\mathbf{\Sigma} = \sqrt{\frac{P_T}{\lambda_2 + \alpha_0\lambda_1}} \begin{bmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\alpha_0\lambda_1} \end{bmatrix},$$

$$\mathbf{V}^H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{e}^{\mathrm{i}\pi/4} \\ \mathrm{e}^{-\mathrm{i}\pi/4} & 1 \end{bmatrix},$$

where $\alpha_0 = 3 - 2\sqrt{2} \approx 1.716 \cdot 10^{-1}$.

Proof: See Appendix I. ■

Noteworthy, the inner structure of the optimal transmission matrix $\mathbf{B}^\star$ depends on the relation of the two largest eigenvalues of $\mathbf{R_H}$.

On one hand, when this relation is low, $\lambda_2/\lambda_1 < \lambda_c$, only the strongest eigenmode is found useful for transmission (the rank of $\mathbf{\Sigma}$ is one), and then a new constellation is created using the two QPSK streams. This new constellation is very similar to a 16-QAM modulation, with little perturbations over the positions of the constellation points in order to maximize the minimum distance among them and, consequently, maximize the number of neighbors at minimum distance. In Fig. 2 this new constellation is depicted.

On the other hand, if the relation λ_2/λ_1 is bigger than the threshold value λ_c, then the two eigenmodes associated with the two largest eigenvalues are used for transmission. In this case, two similar signal constellations with 16 points are transmitted through the two eigenmodes. If we consider the received constellation $\mathbf{HBs}$, we can express it, up to a unitary transformation $\mathbf{Q} \in \mathbb{C}^{n_R \times 2}$ which preserves the distances, as

$$\mathbf{HBs} = \mathbf{Q}\sqrt{\frac{P_T\lambda_1\lambda_2}{\lambda_2 + \alpha_0\lambda_1}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\alpha_0} \end{bmatrix} \mathbf{V}^H\mathbf{s}, \quad (6)$$

where we have represented the two largest eigenmodes of the channel matrix $\mathbf{H}$ in its singular value decomposition, $\mathbf{H} = \mathbf{Q\Lambda}^{\frac{1}{2}}\mathbf{U}_{\mathbf{H}}^H$. See Fig. 3 for a graphical representation of the received constellation. Note that the constellation points at the two different eigenmodes are paired, in the sense that they are transmitted and received together, giving a total of 16 different symbols. This implies that the symbol rate is the same as in the case of using only one channel eigenmode. In addition, the points in the outer circle in one of the eigenmodes are paired with the points in the inner circle in the other eigenmode, and vice versa. Moreover, equation (6) implies that the relation between the sizes of the constellations received through the two eigenmodes is fixed and equal to $\sqrt{\alpha_0} = \sqrt{2} - 1$. This fixed ratio is optimal in the sense that it maximizes the minimum distance between the received constellation points.

To give a visual idea of the two optimal transmission schemes we present a picture of both in Fig. 4.

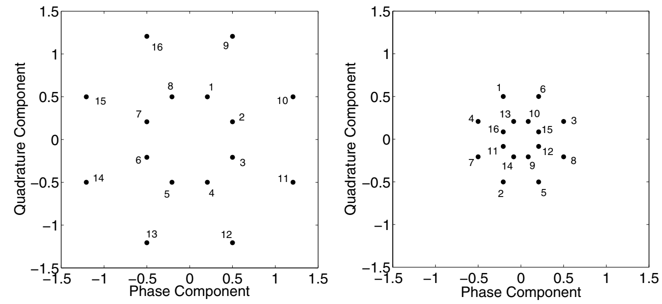

Fig. 3. Received constellation through the two largest eigenmodes. Left, eigenmode associated with λ_1. Right, eigenmode associated with λ_2. Notice that the constellation points at the two different eigenmodes are paired by numbers from 1 to 16. Each pair is transmitted and received together.

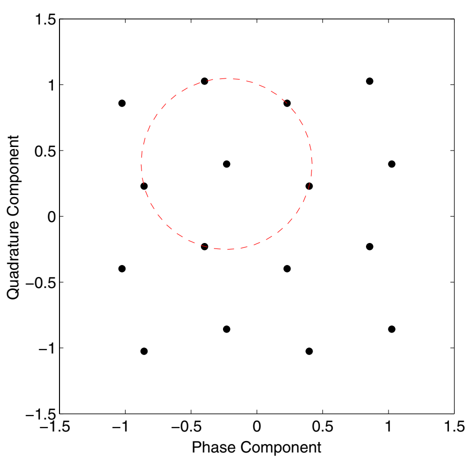

Fig. 2. Optimal received constellation when $\lambda_2/\lambda_1 < \lambda_c$. Notice that the four innermost points have five neighbors at minimum distance, as opposed to the 16-QAM constellation where they have only four neighbors.

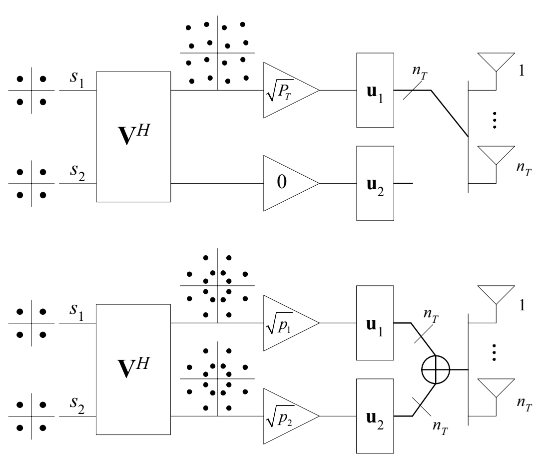

Fig. 4. Graphical representation of the two optimal transmission architectures. The upper scheme is optimal when $\lambda_2/\lambda_1 < \lambda_c$, notice that only one eigenmode is used. The lower scheme is optimal when $\lambda_2/\lambda_1 \geq \lambda_c$.

V. Simulation Results

The transmitter design presented in the previous section is optimal in the sense that it maximizes the minimum distance (MMD) between the received constellation points. To validate the goodness of this criterion, we have compared the performance of our transmission scheme – MMD – with the performance of four well known transmission architectures: BLAST [14], the Alamouti scheme [15], the minimum BER optimal linear design (MBOL) [5], and the maximum minimum SNR eigenvalue design (MMS) [4]. For the sake of fairness, in all these cases the receiver performs ML detection, and the rate is fixed at 4 bit per channel use. This implies that the BLAST scheme transmits two QPSK symbols per channel use, the Alamouti scheme transmits two 16-QAM symbols each two channel uses, and the MMD, MBOL, and MMS designs transmit a linear combination of two QPSK symbols.

We have considered a random 2×2 MIMO channel, with i.i.d. Rayleigh entries. Firstly, we have obtained the pdfs of the squared minimum distance, $d^2_{\min}$, between the received constellation points, when the transmission power is fixed to unity (see Fig. 5). The mean of the pdfs of the squared minimum distance is summarized in Table I. As expected, the MMD scheme presents the highest mean squared minimum distance, because it yields the maximum minimum distance for each channel realization.

TABLE I

Mean $d^2_{\min}$ for Different Transmission Architectures

Scheme	Mean $d^2_{\min}$
Maximum Minimum SNR Eigenvalue (MMS)	0.8001
16-QAM Alamouti	0.8003
QPSK BLAST	0.9924
Optimum Linear (MBOL)	1.0848
Maximum Minimum Distance (MMD)	1.8696

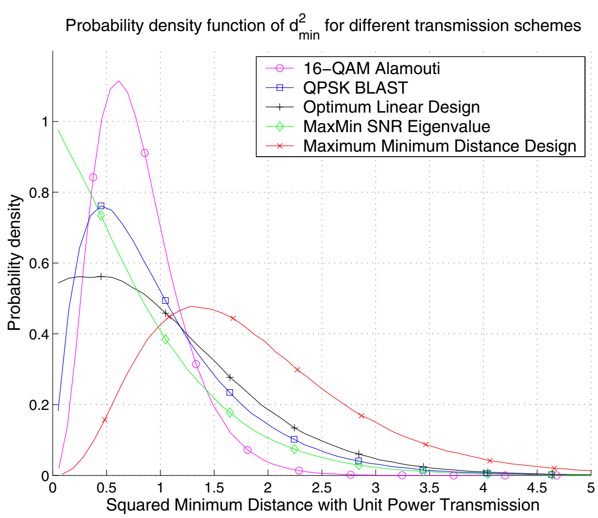

Fig. 5. Probability density function of the squared minimum distance $d^2_{\min}$ for different types of communication schemes. We have fixed $P_T = 1$.

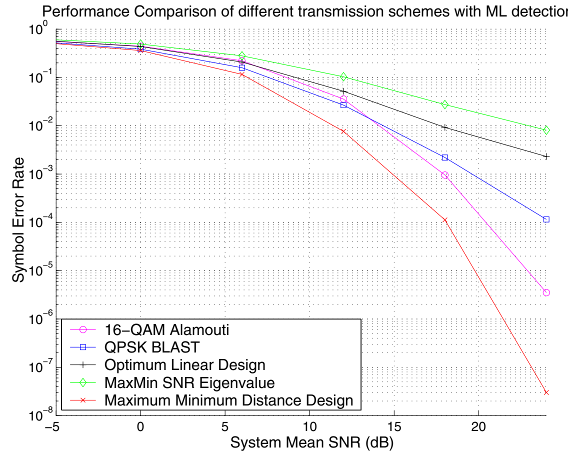

Fig. 6. Symbol error rate for different types of communication schemes.

To complete the comparison, in Fig. 6 we present the mean symbol error rate (SER) vs. the system mean SNR. Also in this case, the MMD scheme shows the best performance. Noteworthy, although the MMD scheme is not originally designed to minimize the mean SER (but the worst-case PEP), it also presents an excellent performance in its terms. This implies that the minimization of the worst-case PEP is a key point in the design of good transmission schemes where the performance is dictated by the SER. In addition, the poor performance of MBOL and MMS, specially at high SNR, is due to the fact that, for these two cases, $d^2_{\min}$ can take values close to zero with non-zero probability (see Fig. 5), as opposed to the other cases (BLAST, Alamouti, and MMD) where the probability that $d^2_{\min}$ takes small values tends to zero.

VI. Discussion and Conclusion

In this work, we have presented the design, in closed form, of the optimal linear transmitter for a MIMO system with full CSI for the particular case of transmitting two QPSK data streams when the objective is the maximization of the minimum distance between all the pairs of received constellation points. We have also found that our proposed scheme yields an excellent performance in terms of mean SER.

In addition, the expression found for the optimal transmitter is computationally simple to calculate: only the svd of the channel matrix is needed and the remaining operations are straightforward. This complexity is the same as needed to compute the MBOL or MMS designs, but it is higher than the simplicity of the BLAST or Alamouti schemes.

Finally, the extension of this problem to other kind of input symbols or number of streams seems much more complicated, *i.e.*, it appears to be a nonscalable problem. The main reason is that the formulated maximin problem is discrete.

Acknowledgements

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APPENDIX I

Here we present a sketch of the proof of Proposition 1. For a complete proof, see further [16].

A. Two Transmitters Case, $n_T = 2$

In this first part of the proof, we consider that $\mathbf{H} \in \mathbb{C}^{n_R \times 2}$. Consequently, $\mathbf{R_H} \in \mathbb{C}^{2\times 2}$ will be assumed.[1] We express the quadratic form in (5) as $\mathbf{e}^H\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^H\mathbf{Q_H}\boldsymbol{\Lambda}\mathbf{Q_H}^H\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\mathbf{e}$, where $\mathbf{B} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$, and $\mathbf{R_H} = \mathbf{Q_H}\boldsymbol{\Lambda}\mathbf{Q_H}^H$ have been used and where $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{Q_H}$ are unitary matrices and $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}$ are diagonal positive semidefinite matrices. Notice that the restriction $\mathrm{Tr}\,\mathbf{B}\mathbf{B}^H \le P_T$ becomes $\mathrm{Tr}\,\boldsymbol{\Sigma}\boldsymbol{\Sigma}^H \le P_T$. Since $\mathbf{Q_H}$ is known, we define $\widetilde{\mathbf{U}} = \mathbf{Q_H}^H\mathbf{U}$ as a new optimization variable. We obtain the new quadratic form $\mathbf{e}^H\mathbf{V}\boldsymbol{\Sigma}\widetilde{\mathbf{U}}^H\boldsymbol{\Lambda}\widetilde{\mathbf{U}}\boldsymbol{\Sigma}\mathbf{V}^H\mathbf{e}$.

Now, suppose that we fix $\widetilde{\mathbf{U}}$ and $\boldsymbol{\Sigma}$. We can then express the positive semidefinite matrix $\boldsymbol{\Sigma}\widetilde{\mathbf{U}}^H\boldsymbol{\Lambda}\widetilde{\mathbf{U}}\boldsymbol{\Sigma}$ as $\boldsymbol{\Sigma}\widetilde{\mathbf{U}}^H\boldsymbol{\Lambda}\widetilde{\mathbf{U}}\boldsymbol{\Sigma} = \mathbf{Q_L}\mathbf{L}\mathbf{Q_L}^H$, where $\mathbf{Q_L}$ and $\mathbf{L}$ are functions of $\widetilde{\mathbf{U}}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\Lambda}$ and where $\mathbf{Q_L}$ is unitary and $\mathbf{L}$ is diagonal. Repeating the same trick as before we now define $\widetilde{\mathbf{V}} = \mathbf{V}\mathbf{Q_L}$, and then the original problem in (5) becomes

$$\max_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \max_{\widetilde{\mathbf{V}}} \min_{\mathbf{e}} \quad \mathbf{e}^H\widetilde{\mathbf{V}}\mathbf{L}\widetilde{\mathbf{V}}^H\mathbf{e},$$

where the dependence of the objective function on $\widetilde{\mathbf{U}}$ and $\boldsymbol{\Sigma}$ is implicit in $\mathbf{L}$. Since $\mathbf{L}$ is a diagonal matrix and $\widetilde{\mathbf{V}}$ is unitary, they can be parameterized as

$$\mathbf{L} = \begin{bmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{bmatrix}, \text{ and } \widetilde{\mathbf{V}} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\delta}\cos\theta & \mathrm{e}^{\mathrm{i}\epsilon}\sin\theta \\ -\mathrm{e}^{-\mathrm{i}\epsilon}\sin\theta & \mathrm{e}^{-\mathrm{i}\delta}\cos\theta \end{bmatrix}.$$

The optimization problem becomes

$$\max_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \max_{\theta,\delta,\epsilon} \min_{q\le|\mathcal{E}|} \quad \mathbf{e}_q^H\widetilde{\mathbf{V}}\mathbf{L}\widetilde{\mathbf{V}}^H\mathbf{e}_q,$$

where now the error vector has been indexed as explained in Section III. For the sake of notation, let us define the objective function for the possible error vectors as $f_q \equiv \mathbf{e}_q^H\widetilde{\mathbf{V}}\mathbf{L}\widetilde{\mathbf{V}}^H\mathbf{e}_q$, with $q \le |\mathcal{E}|$. Since there are elements in the set $\mathcal{E}$ that are proportional, *i.e.*, $\mathbf{e}_r = C\mathbf{e}_s$, $C \in \mathbb{C}$, it is possible to discard 66 elements from the total of 80 in $\mathcal{E}$ because either they yield the same objective function ($|C| = 1, f_r = f_s$) or because one of the objective functions is always greater ($|C| > 1, f_r > f_s$). Consequently, we end up with 14 different objective functions, $f_1, \ldots, f_{14}$. The explicit expressions for these functions can be found in [16]. We just give here three of them, which are the most significative in this sketch of the proof. They are

$$\begin{aligned} f_1 &= 2\left(\ell_1\sin^2\theta + \ell_2\cos^2\theta\right), \\ f_2 &= 2\left(\ell_1 + \ell_2 + 2(\ell_2 - \ell_1)\sin\theta\cos\theta\cos\phi\right), \\ f_3 &= 2(\ell_1 + \ell_2 + \ell_1\sin^2\theta + \ell_2\cos^2\theta + \\ &\quad + 2\sqrt{2}(\ell_2 - \ell_1)\sin\theta\cos\theta\cos(\phi - \pi/4)), \end{aligned}$$

where $\phi = \delta + \epsilon$. Since the only dependence of $\{f_q\}$ on δ and ϵ is through ϕ, the problem now becomes

$$\max_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \max_{\theta,\phi} \min_{q\le 14} \quad f_q(\theta,\phi).$$

[1] As a consequence of $\mathbf{R_H} \in \mathbb{C}^{2\times 2}$, all the other matrices that appear in the following of this Section I-A also belong to the set $\mathbb{C}^{2\times 2}$.

In principle, the search space for the inner maximization part is $(\theta,\phi) \in [-\pi,\pi]\times[-\pi,\pi]$, but, from the specific dependence of $\{f_q\}$ on (θ,ϕ) it can be shown that the search space can be reduced to $(\theta,\phi) \in [0,\pi/4]\times[0,\pi/4] \equiv \mathcal{D}$.

Without loss of generality we will assume that $\ell_1 \ge \ell_2$. In this case, one readily sees that

$$(\theta,\phi) \in \mathcal{D} \Rightarrow \min_{q\le 3} f_q(\theta,\phi) \le \min_{4\le q\le 14} f_q(\theta,\phi).$$

Finally, we obtain an equivalent formulation of (5) as

$$\max_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \max_{(\theta,\phi)\in\mathcal{D}} \min_{q\le 3} \quad f_q(\theta,\phi). \tag{7}$$

Let us define $\alpha \equiv \ell_2/\ell_1 \in [0,1]$, and $\alpha_0 \equiv 3 - 2\sqrt{2}$. We can distinguish two different cases.

- $\alpha \in [\alpha_0, 1] \equiv \mathcal{A}_1$: The two innermost optimization problems in (7) can be reduced to
$$\max_{(\theta,\phi)\in\mathcal{D}} \min_{q\le 3} f_q(\theta,\phi) = \max_{(\theta,\phi)\in\mathcal{D}} f_1(\theta,\phi) = \ell_1 + \ell_2,$$
where the maximum is attained for $\theta^\star = \phi^\star = \pi/4$.
- $\alpha \in [0, \alpha_0) \equiv \mathcal{A}_0$: In this case, the two innermost problems in (7) can be bounded by
$$\max_{(\theta,\phi)\in\mathcal{D}} \min_{q\le 3} f_q \le K(\lambda_c\ell_1 + \ell_2), \tag{8}$$
where $\lambda_c \equiv \dfrac{(\sqrt{3}-1)(3-2\sqrt{2})}{1+3\sqrt{3}-2\sqrt{6}}$, and $K \equiv \dfrac{1+\alpha_0}{\lambda_c+\alpha_0}$.
The general expressions for $\theta^\star$ and $\phi^\star$ in this case are not reproduced here for the sake of space, because they depend on the remaining variables $\widetilde{\mathbf{U}}$ and $\boldsymbol{\Sigma}$.

This last two cases allow us to split the original problem in (7) into two simpler problems. Namely,

$$\max_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \max_{(\theta,\phi)\in\mathcal{D}} \min_{q\le 3} f_q \begin{cases} = \max\limits_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} \ell_1 + \ell_2 & \alpha \in \mathcal{A}_1 \\ \le \max\limits_{\widetilde{\mathbf{U}},\boldsymbol{\Sigma}} K(\lambda_c\ell_1 + \ell_2) & \alpha \in \mathcal{A}_0 \end{cases}, \tag{9}$$

where the dependence of ℓ_1 and ℓ_2 on $\widetilde{\mathbf{U}}$ and $\boldsymbol{\Sigma}$ is through the eigenequation $\boldsymbol{\Sigma}\widetilde{\mathbf{U}}^H\boldsymbol{\Lambda}\widetilde{\mathbf{U}}\boldsymbol{\Sigma} = \mathbf{Q_L}\mathbf{L}\mathbf{Q_L}^H$, where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$
$$\text{and } \widetilde{\mathbf{U}} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\chi}\cos\psi & \mathrm{e}^{\mathrm{i}\eta}\sin\psi \\ -\mathrm{e}^{-\mathrm{i}\eta}\sin\psi & \mathrm{e}^{-\mathrm{i}\chi}\cos\psi \end{bmatrix}, \tag{10}$$

with $\sigma_1^2 + \sigma_2^2 \le P_T$. We define $\gamma = \sigma_1^2 + \sigma_2^2$, $p = \sigma_1^2/\gamma \in [0,1]$, and $\lambda = \lambda_2/\lambda_1 \le 1$. Solving the eigenequation $\boldsymbol{\Sigma}\widetilde{\mathbf{U}}^H\boldsymbol{\Lambda}\widetilde{\mathbf{U}}\boldsymbol{\Sigma} = \mathbf{Q_L}\mathbf{L}\mathbf{Q_L}^H$, we obtain an expression for $\ell_{1,2}$

$$\begin{aligned} b &= (p + \lambda - \lambda p)\cos^2\psi + (1 - p + \lambda p)\sin^2\psi, \\ \ell_{1,2} &= \gamma\lambda_1\frac{b \pm \sqrt{b^2 - 4\lambda p(1-p)}}{2}. \end{aligned} \tag{11}$$

Notice that $\ell_{1,2}$ do not depend on χ nor η, as defined in (10). We will assume $\chi = \eta = 0$ without loss of generality. Consequently the two optimization problems in (9) can be further simplified. When $\alpha \in \mathcal{A}_1$ we obtain

$$\max_{p,\psi} \gamma\lambda_1 b, \tag{12}$$

and when $\alpha \in \mathcal{A}_0$ the bound becomes, up to a constant factor,

$$\max_{p,\psi} (\lambda_c+1)b + (\lambda_c-1)\sqrt{b^2-4\lambda p(1-p)}, \quad (13)$$

where b in (12) and (13) is the same as defined in (11). In [16], it is shown that $\psi^\star = 0$ is an optimal solution for the two problems in (12) and (13). Thus, the problem in (12) reduces to

$$\max_p \quad p+\lambda-\lambda p = p(1-\lambda)+\lambda \quad (14)$$

$$\text{s.t.} \quad p \in \left[\frac{\lambda}{\lambda+1/\alpha_0}, \frac{\lambda}{\lambda+\alpha_0}\right], \quad (15)$$

where the restriction on p is due to the fact that $\alpha \in \mathcal{A}_1$ must be guaranteed. Since the factor $(1-\lambda)$ in the objective function in (14) is bigger than zero, the optimum will be attained when p equals its maximum value, *i.e.*, $p^\star = \frac{\lambda}{\lambda+\alpha_0}$.

Similarly, with $\psi^\star = 0$, the problem in (13) is simplified to

$$\max_p \quad K\begin{cases} (1-\lambda_c\lambda)p + \lambda_c\lambda & \text{for } p < \frac{\lambda}{1+\lambda} \\ (\lambda_c-\lambda)p+\lambda & \text{for } p \geq \frac{\lambda}{1+\lambda} \end{cases},$$

$$\text{s.t.} \quad p \in \left[0, \frac{\lambda}{\lambda+1/\alpha_0}\right) \cup \left(\frac{\lambda}{\lambda+\alpha_0}, 1\right], \quad (16)$$

where, in this case the restriction on p is to guarantee that $\alpha \in \mathcal{A}_0$.[2] The solution of the maximization depends on a condition on λ. On one hand, if $\lambda < \lambda_c$, then the optimum is $p^\star = 1$. On the other hand, if $\lambda \geq \lambda_c$, the optimum is attained by $p^\star \to \frac{\lambda}{\lambda+\alpha_0}$, which coincides with the optimum solution for the problem in (12) and which implies that, when $\lambda \geq \lambda_c$, the solution lies in the boundary of the two regions, *i.e.*, $\alpha = \alpha_0$. Finally, we recall that what we obtained is just the maximization of the bound in (8), in [16] it is checked that the bound is actually attained by the objective function.

Now that we have obtained the optimal values for all the parameters, $\theta^\star$, $\phi^\star$, $\psi^\star$, and $p^\star$, it remains to evaluate the objective function in (9), for the two cases: $\alpha \in \mathcal{A}_0$ and $\alpha \in \mathcal{A}_1$ and calculate in each case which function yields the highest value. By doing so (see further [16]), we obtain:

- $\lambda = \lambda_2/\lambda_1 < \lambda_c$: The optimal values of the variables are

$$\theta^\star = \arctan\left(\sqrt{6}/(3+\sqrt{3})\right), \qquad \psi^\star = 0,$$
$$\phi^\star = \arccos\left((3+\sqrt{3})/(2\sqrt{6})\right), \qquad p^\star = 1,$$

  where $\theta^\star$ and $\phi^\star$ are such that the bound in (8) is attained for $\psi = 0$ and $p = 1$. In this case $\alpha = 0$.

- $\lambda_2/\lambda_1 \geq \lambda_c$: The optimal values of the variables are

$$\theta^\star = \pi/4, \qquad \psi^\star = 0,$$
$$\phi^\star = \pi/4, \qquad p^\star = \lambda/(\lambda+\alpha_0),$$

  where in this case $\alpha = \alpha_0$.

Notice that in both cases we have obtained that $\psi^\star = 0$ which means that $\widetilde{\mathbf{U}}^\star = \mathbf{I}$ or, equivalently $\mathbf{U}^\star = \mathbf{Q_H}$. In addition, last statement implies that $\mathbf{Q_L} = \mathbf{I}$ and consequently $\widetilde{\mathbf{V}} = \mathbf{V}$.

[2] Notice that the two intervals of definition of p in (15) and (16) do not overlap, and its union is the interval $[0,1]$.

B. Generalization to any number of transmitters

For the case $n_T > 2$, the objective function in (5) becomes

$$\mathbf{e}^H\mathbf{V}\mathbf{\Sigma}\mathbf{U}^H\mathbf{Q_H}\mathbf{\Lambda}\mathbf{Q_H}^H\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{e}, \quad (17)$$

where now $\mathbf{Q_H} \in \mathbb{C}^{n_T\times n_T}$ and $\mathbf{U} \in \mathbb{C}^{n_T\times 2}$. From the Poincaré Separation Theorem [17], the two eigenvalues of $\mathbf{U}^H\mathbf{Q_H}\mathbf{\Lambda}\mathbf{Q_H}^H\mathbf{U}$, which we denote by $\mu_1 \geq \mu_2$, must fulfill

$$\mu_1 \leq \lambda_1, \quad \mu_2 \leq \lambda_2, \quad (18)$$

where λ_i are the diagonal elements of $\mathbf{\Lambda}$ in decreasing order. Both equalities in (18) hold when $\mathbf{U}$ has as columns the two columns of $\mathbf{Q_H}$ with the eigenvectors associated to the two largest eigenvalues λ_1 and λ_2. This matrix is denoted by $\mathbf{U_H}$. For the $n_T > 2$ case, $\mathbf{U}^\star = \mathbf{U_H}$ must hold, because the expression in (17) is an increasing function on μ_1 and μ_2.

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