Robust Power Allocation Designs for Multiuser and Multiantenna Downlink Communication Systems through Convex Optimization

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***Abstract*— In this paper, we study the design of the transmitter in the downlink of a multiuser and multiantenna wireless communications system, considering the realistic scenario where only an imperfect estimate of the actual channel is available at both communication ends. Precisely, the actual channel is assumed to be inside an uncertainty region around the channel estimate, which models the imperfections of the channel knowledge that may arise from, *e.g.*, estimation Gaussian errors, quantization effects, or combinations of both sources of errors. In this context, our objective is to design a robust power allocation among the information symbols that are to be sent to the users such that the total transmitted power is minimized, while maintaining the necessary quality of service to obtain reliable communication links between the base station and the users for any possible realization of the actual channel inside the uncertainty region. This robust power allocation is obtained as the solution to a convex optimization problem, which, in general, can be numerically solved in a very efficient way, and even for a particular case of the uncertainty region, a quasi-closed form solution can be found. Finally, the goodness of the robust proposed transmission scheme is presented through numerical results.**

***Index Terms*— Robust designs, imperfect CSI, multiantenna systems, broadcast channel, convex optimization.**

I. Introduction

MULTIANTENNA techniques for single user communications have been proved to provide multiplexing and diversity gains, even simultaneously under the tradeoff described in [1]. In general, the gains that can be achieved depend, to a large extent, on the quantity and quality of the channel state information (CSI) that is available at the transmitter and/or receiver ends.

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Although the benefits for single user communications have been studied extensively, one of the main potentials of multiantenna communications is that they can afford multiuser communications, where the different signals can be separated by spatial processing techniques. As opposed to single user communications, in the design of multiuser systems, several quality of service (QoS) measures have to be considered simultaneously, each one corresponding to each user. This leads to an inherent problem in the design of such systems, which is how to handle with these different quality measures.

A possible simplified solution, inspired by single user designs, aims at optimizing the mean value of these measures [2], [3]. However, the main drawback of these strategies, is that the optimization of the mean does not guarantee a minimum acceptable quality for all the data streams. Consequently, a more suitable approach, is to guarantee a minimum QoS independently for the data stream corresponding to each user, while optimizing a global network parameter such as the total transmitted power.

Similar to the single user case, the performance in multiuser communications also depends on the available CSI. Initially, most researchers concentrated their efforts on the design of transmission architectures assuming that both the transmitter and the user receivers have perfect knowledge of the CSI, giving rise to the so-called solution of the downlink beamforming problem, [4], [5]. Very recently, a unified framework with a very powerful and general model to deal with the problem of power allocation design in a multiuser and multiantenna downlink scenario with perfect CSI has been presented in [6].

Note, however, that in a realistic implementation of the system, the assumption of the availability of a complete and perfect CSI is too optimistic due to the fluctuations of the channel and also due to the presence of estimation errors or quantization effects in the CSI. In the case of imperfect CSI, the simplest approach consists in utilizing the available CSI as if it was perfect, giving rise to naive (non-robust) designs. It has been shown that these designs are extremely sensitive to the errors in the CSI [7], [8], which translates into a decrease of the system performance.

Thus, the optimal approach in this case is to consider a robust design where the presence of the errors in the CSI is explicitly taken into consideration. There are essentially two different ways of doing this, depending on the model assumed for the errors. On one hand, in the Bayesian philosophy the errors are modeled from a statistical point of view as in [9]–[11], and, on the other hand, in the maximin approach a statistical description of the error is not needed, because it is assumed that the error belongs to a predefined uncertainty region, whose shape and size are linked to the physical phenomenon producing the error in the CSI as in [11]–[16]. In general, the design of a robust technique is much more complicated than its non-robust counterpart. This is the reason why in many of the papers dealing with this topic, some kind of simplification has to be imposed in the system architecture (for example, even for the single user case in [17]–[19] the particular case of concatenating an orthogonal space time block code (OSTBC) and a linear transform was investigated).

In this paper, we consider the downlink of a multiuser communications system with several single antenna receivers and a multiantenna base station. The transmitter is composed of two blocks: a power allocation among the symbols for different users, and a linear transformation. The robustness of our system is achieved by a maximin design of the power allocation under two considerations. On one hand, the objective is to minimize the total transmitted power, and, on the other hand, we wish to guarantee a certain minimum QoS per user for any possible error of the CSI inside the uncertainty region. The design of the power allocation fulfilling these two considerations is formulated as a convex optimization problem, which is next solved for several uncertainty regions, modeling the most practical cases of errors in the CSI: estimation Gaussian errors and/or quantization effects (see [20] for a complete study of the capacity degradation due to quantization effects). The main advantage of formulating our optimization problem within the convex optimization framework is that numerical solutions can be computed very efficiently, and, even for a particular case, a quasi-closed form solution can be found.

To the best of our knowledge, the existing literature dealing with maximin transmitter design in multiuser systems with imperfect CSI is [4], [21], [22], and some references therein by the same authors. The main differences from these works and ours are that they assume that an imperfect estimate of the channel *covariance* matrix is made available at the transmitter, and that the only considered uncertainty region for this error is a spherical uncertainty region.

The remainder of the paper is organized as follows. In the next section we present the system model of our downlink wireless communication system. In Section III we analyze the effects of having imperfect CSI and formally state the general formulation for the problem of finding the power allocation that minimizes the total transmitted power while guaranteeing a minimum QoS per user. This general problem is particularized in Section IV for a family of uncertainty regions. In Section V we discuss some practical aspects regarding the calculation of the numerical solution of the obtained convex optimization problems. Finally, in Section VI some numerical examples of the techniques presented are shown and in Section VII the conclusions are drawn.

II. SYSTEM MODEL

We consider the downlink of a multiuser communications system, where the base station utilizes n_T antennas to simultaneously transmit information symbols to n_U users with single antenna terminals. The baseband model for the samples of the received signal by i-th user is

$$y_i = \mathbf{h}_i^H \mathbf{x} + n_i, \quad i = 1, \ldots, n_U, \tag{1}$$

where $\mathbf{h}_i^H \in \mathbb{C}^{1\times n_T}$ is the flat fading spatial channel response from the n_T transmission antennas to the i-th user, $\mathbf{x} \in \mathbb{C}^{n_T\times 1}$ represents the transmitted signal by the base station through all the antennas, and n_i is the noise contribution with $\mathbb{E}|n_i|^2 = \sigma^2$, $\forall i$. A more compact expression is obtained by stacking the received signals, y_i, and the noise components, n_i, into the column vectors $\mathbf{y} \in \mathbb{C}^{n_U\times 1}$ and $\mathbf{n} \in \mathbb{C}^{n_U\times 1}$, and by defining $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_{n_U}]^H \in \mathbb{C}^{n_U\times n_T}$. With these definitions, the received signals vector can be expressed as $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$.

The transmitted signal $\mathbf{x}$ is designed as a linear function of the information symbols vector, $\mathbf{s} \in \mathbb{C}^{n_U}$, where s_i represents the symbol to be communicated to i-th user and where $\mathbb{E}\mathbf{s}\mathbf{s}^H = \mathbf{I}_{n_U}$ is assumed w.l.o.g. This linear combination is expressed as the product of two linear transformations as

$$\mathbf{x} = \mathbf{B}\mathbf{P}^{1/2}\mathbf{s}, \tag{2}$$

where $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_{n_U}]$ is the transmission matrix and the diagonal matrix $\mathbf{P}^{1/2}$, with elements $[\mathbf{P}^{1/2}]_{ii} = \sqrt{p_i}$, takes into account the power allocation among the information symbols.

As commented in the introduction, the objective in this work is to design the transmitter according to the available information about the actual channel matrix at both communications ends, which is assumed to be imperfect. In order to optimize the global system performance, the presence of these imperfections has to be taken into account explicitly, leading to robust solutions that are less sensitive to these errors. The mathematical problem arising from the robust design of the whole transmitter $\mathbf{B}\mathbf{P}^{1/2}$ is generally much more complicated than the classical non-robust solution. This too demanding complexity requires to make some assumptions and simplifications in the design, as seen in many works such as [4], [5], [17]–[19] and also in this paper (indeed, these simplifications may be required not only to solve the mathematical problem itself, but also to obtain a solution that can be implemented in a realistic system with restrictions on the allowed computational load). Concretely, in our case, the design of the transmitter is simplified by dividing it into two parts taking an engineering and practical perspective. The transmission matrix $\mathbf{B}$ is allowed to depend only on the channel estimate and it is designed in a non-robust way according to a predefined performance criterion. On the other hand, the design of the power allocation $\mathbf{P}^{1/2}$ is much more general and is allowed to depend not only on the channel estimate, but also on the model of the imperfections in the CSI. In other words, the robustness is achieved through the addition of a power allocation block before the symbols are processed by the matrix $\mathbf{B}$ as depicted in Fig. 1. Note then, that the focus of this paper is on the power allocation itself. Indeed, this focus on this transmitter separation into two blocks has also been taken in excellent works such as [5], [6], and references therein by the same authors, where the power allocation is designed assuming perfect CSI, *i.e.*, they do not analyze the robustness problem. As commented before, a similar work

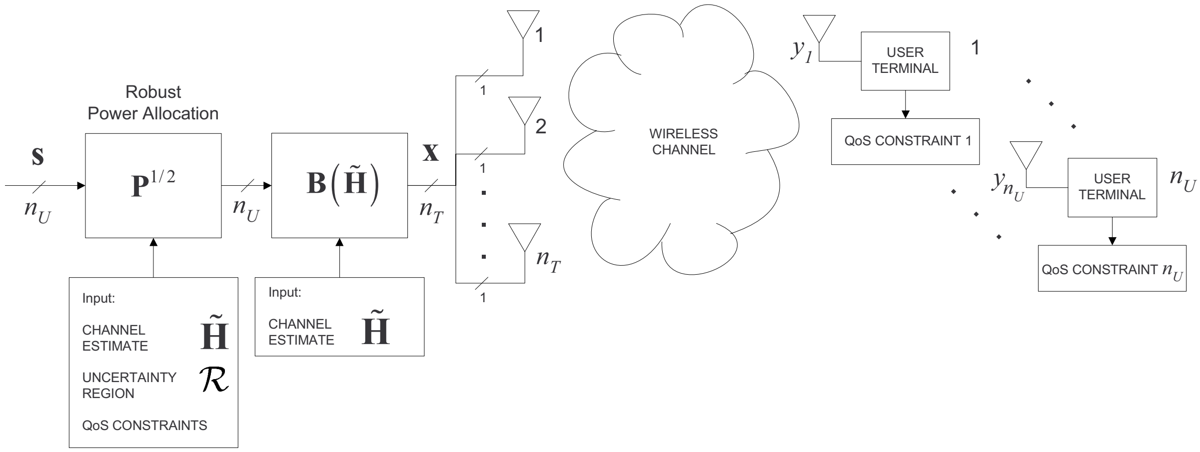

Fig. 1. Robust downlink multiuser communication scheme. Note that there is a power allocation block at the transmitter, that weights the information symbols prior to transforming them with the linear filter $\mathbf{B}$. The power allocation is robustly designed to minimize the transmitted power while guaranteeing a certain QoS for each user. The optimal power allocation is found as a function of the channel estimate $\widetilde{\mathbf{H}}$, the uncertainty region $\mathcal{R}$, and the set of QoS constraints $\{\mathsf{qos}_i^0\}$.

as the one presented here on the design of a robust power allocation has been conducted in [22], where, as in this paper, the focus is not given to the design of the linear transmission matrix $\mathbf{B}$, either.

III. Imperfect Channel State Information and Problem Statement

In a practical communications scenario, the assumption of perfect CSI at the transmitter and receiver sides is rather unrealistic. At the receivers side, the channel is usually estimated through training sequences (pilot symbols), and at the transmitter side, the CSI can be acquired through a feedback channel in FDD systems or from previously received symbols by exploiting the channel reciprocity in TDD systems. In both cases, different sources of errors can be identified depending on how the CSI is obtained, such as estimation Gaussian errors or quantization effects.

In this section, we analyze the case where, due to the aforementioned imperfection in the CSI, both the transmitter and the receiver have only access to the *same* imperfect estimate, $\widetilde{\mathbf{H}} = [\tilde{\mathbf{h}}_1, \ldots, \tilde{\mathbf{h}}_{n_U}]^H$, of the actual channel $\mathbf{H}$, which is considered to obtain a tractable problem. In some situations this corresponds to practical scenarios (as described in the following sections). In other cases, it is possible that the receiver has a *better* estimate of the channel than the transmitter (in these cases our approach yields a lower bound on the performance that could be achieved by the system). In addition, we assume that the actual channel is inside an uncertainty region, $\mathcal{R} \subset \mathbb{C}^{n_U \times n_T}$, around its estimate similarly as in [12], [14]–[16], which yields

$$\mathbf{H} = \widetilde{\mathbf{H}} + \boldsymbol{\Delta}, \tag{3}$$

for some $\boldsymbol{\Delta} = [\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{n_U}]^H \in \mathcal{R}$. The shape and size of $\mathcal{R}$ model the kind of uncertainty in the channel estimate. For example, if the uncertainty stems from the fact that the CSI is a uniformly quantized version of the actual channel, then the entries of the error matrix are inside the interval $[\boldsymbol{\Delta}]_{ij} \in [-\rho, \rho] \times [-\mathrm{j}\rho, \mathrm{j}\rho]$, where 2ρ is the quantization step, and thus the uncertainty region is a hypercube, whose side length is the quantization step (more details are given in Section IV or see further [16]).

As mentioned in the introduction, given a design for the transmitter matrix $\mathbf{B}$ as a function of the channel estimate $\widetilde{\mathbf{H}}$, the most general formulation of our problem is to robustly design the power allocation matrix $\mathbf{P}$, as in (4), so that the total transmitted power, $P_T(\mathbf{P})$, is minimized, and the QoS for every user, $\mathsf{qos}_i(\mathbf{P}, \boldsymbol{\Delta})$, is always above a fixed minimum quality threshold, qos_i^0, for any possible realization of the error matrix $\boldsymbol{\Delta}$ inside the uncertainty region $\mathcal{R}$.

$$\begin{aligned} \underset{\mathbf{P}}{\text{minimize}} \quad & P_T(\mathbf{P}), \\ \text{subject to} \quad & \mathsf{qos}_i(\mathbf{P}, \boldsymbol{\Delta}) \geq \mathsf{qos}_i^0, \quad p_i \geq 0, \\ & \qquad \forall i \in \{1, 2, \ldots, n_U\}, \quad \forall \boldsymbol{\Delta} \in \mathcal{R}. \end{aligned} \tag{4}$$

Note that, by explicitly taking into account the imperfections of the CSI in the design process we obtain a communications system which is robust to uncertainties in the channel estimate. The robustness of the proposed system stems from the fact that, by explicitly guaranteeing that the QoS for every user is above a certain different threshold for any possible error realization inside the uncertainty region, an increase in the reliability against estimation errors is provided to the users. In the following, when writing particularizations of the general problem in (4) the constraints $p_i \geq 0$ will be dropped for the sake of space, but it is important to remember that they are implicitly assumed.

Since the MSE is a widely utilized performance metric in the existing literature, *e.g.*, [3], [13], the QoS indicator is chosen to be the inverse of the MSE perceived by each user $1/\mathsf{mse}_i$. It is chosen to be the inverse of the MSE because the MSE is a performance metric, which is desired to be as low as possible, and consequently, its inverse is desired to be as high as possible and thus it can be utilized as a QoS indicator.

In the following, we particularize the general optimization problem in (4) for our scheme. From (2), the transmitted power for this architecture is obtained as the linear function

$$P_T(\mathbf{P}) = \mathbb{E}\,\mathrm{Tr}\,\mathbf{x}\mathbf{x}^H = \mathrm{Tr}\,\mathbf{B}\mathbf{P}\mathbf{B}^H. \tag{5}$$

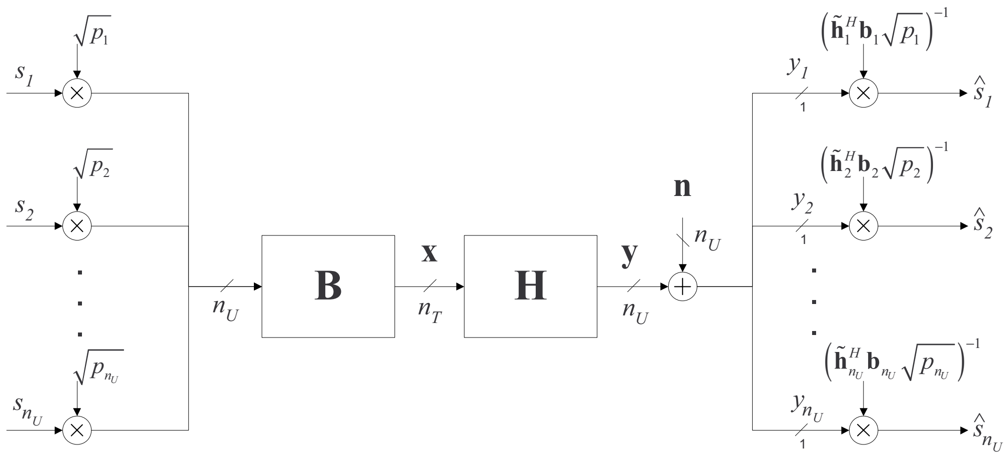

Fig. 2. Schematic representation of the downlink communication system. Each receiver estimates its own symbol $\hat{s}_i$ dividing the incoming signal by the coefficient of s_i in the expression for the received signal in (7).

The expression for the MSE for each user can be obtained as follows. We begin from the received signal for i-th user in (1) when considering the uncertainty model in (3):

$$\begin{aligned} y_i &= \mathbf{h}_i^H \mathbf{B}\mathbf{P}^{1/2}\mathbf{s} + n_i \\ &= \tilde{\mathbf{h}}_i^H \mathbf{B}\mathbf{P}^{1/2}\mathbf{s} + \boldsymbol{\delta}_i^H \mathbf{B}\mathbf{P}^{1/2}\mathbf{s} + n_i \\ &= \tilde{\mathbf{h}}_i^H \mathbf{b}_i \sqrt{p_i} s_i + (\tilde{\mathbf{h}}_i^H \bar{\mathbf{B}}_i + \boldsymbol{\delta}_i^H \mathbf{B})\mathbf{P}^{1/2}\mathbf{s} + n_i, \end{aligned} \tag{6}$$

where $\bar{\mathbf{B}}_i \equiv [\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{0}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{n_U}]$ (see Fig. 2 for a schematic representation of the received signal). Expanding the second term in last equation, we obtain

$$\begin{aligned} y_i = \tilde{\mathbf{h}}_i^H \mathbf{b}_i \sqrt{p_i} s_i + \tilde{\mathbf{h}}_i^H \sum_{j \neq i} \mathbf{b}_j \sqrt{p_j} s_j &+ \\ + \boldsymbol{\delta}_i^H \sum_{j \neq i} \mathbf{b}_j \sqrt{p_j} s_j + \boldsymbol{\delta}_i^H \mathbf{b}_i \sqrt{p_i} s_i + n_i&. \end{aligned} \tag{7}$$

Now it is important to determine precisely the way how each user is going to estimate its own received symbol according to the reception of y_i. We assume that the receivers are constrained to be simple and, consequently, adjust their detection thresholds by dividing their incoming signal by the estimated equivalent channel, which is given by $\tilde{\mathbf{h}}_i^H \mathbf{b}_i \sqrt{p_i}$ obtaining the symbol estimate $\hat{s}_i$ as

$$\hat{s}_i = \frac{y_i}{\tilde{\mathbf{h}}_i^H \mathbf{b}_i \sqrt{p_i}} = s_i + \frac{(\tilde{\mathbf{h}}_i^H \bar{\mathbf{B}}_i + \boldsymbol{\delta}_i^H \mathbf{B})\mathbf{P}^{1/2}\mathbf{s} + n_i}{\tilde{\mathbf{h}}_i^H \mathbf{b}_i \sqrt{p_i}}. \tag{8}$$

The MSE of i-th user is thus given by

$$\begin{aligned} \mathsf{mse}_i &= \mathbb{E}|s_i - \hat{s}_i|^2 = \\ &= \frac{(\tilde{\mathbf{h}}_i^H \bar{\mathbf{B}}_i + \boldsymbol{\delta}_i^H \mathbf{B})\mathbf{P}(\bar{\mathbf{B}}_i^H \tilde{\mathbf{h}}_i + \mathbf{B}^H \boldsymbol{\delta}_i) + \sigma^2}{|\tilde{\mathbf{h}}_i^H \mathbf{b}_i|^2 p_i}. \end{aligned} \tag{9}$$

As it has been said above, although it is formally equivalent, it will be more convenient to consider as a QoS indicator the inverse of the MSE of each user. In this case it is given by

$$\frac{1}{\mathsf{mse}_i} = \frac{|\tilde{\mathbf{h}}_i^H \mathbf{b}_i|^2 p_i}{(\tilde{\mathbf{h}}_i^H \bar{\mathbf{B}}_i + \boldsymbol{\delta}_i^H \mathbf{B})\mathbf{P}(\bar{\mathbf{B}}_i^H \tilde{\mathbf{h}}_i + \mathbf{B}^H \boldsymbol{\delta}_i) + \sigma^2} \triangleq \mathsf{esinr}_i. \tag{10}$$

Note that the structure of the expression in (10) for the inverse of mse_i, is analogous to that of the SINR, so we rename the inverse of the MSE as the *effective* SINR. Although esinr_i is not a true SINR[1], we can use it as a performance metric recalling that it is the inverse of the MSE.

Particularizing the problem in (4) with the expression for the transmitted power in (5) and where the QoS constraint $\mathsf{qos}_i \geq \mathsf{qos}_i^0$ is rewritten with the effective SINR expression in (10), we obtain (note that $\mathsf{esinr}_i^0 = 1/\mathsf{mse}_i^0$):

$$\begin{aligned} &\underset{\mathbf{P}}{\text{minimize}} \quad \operatorname{Tr} \mathbf{B}\mathbf{P}\mathbf{B}^H, \\ &\text{subject to} \quad \sup_{\boldsymbol{\Delta} \in \mathcal{R}} \big(\tilde{\mathbf{h}}_i^H \bar{\mathbf{B}}_i + \boldsymbol{\delta}_i^H \mathbf{B}\big)\mathbf{P}\big(\bar{\mathbf{B}}_i^H \tilde{\mathbf{h}}_i + \mathbf{B}^H \boldsymbol{\delta}_i\big) - \\ &\qquad\qquad - \mathsf{mse}_i^0 \big|\tilde{\mathbf{h}}_i^H \mathbf{b}_i\big|^2 p_i + \sigma^2 \leq 0, \quad \forall i. \end{aligned} \tag{11}$$

Since the restrictions for each user (indexed by i) in the optimization problem (4) have to be guaranteed for all $\boldsymbol{\Delta} \in \mathcal{R}$, they have to be fulfilled, in particular, for the worst-case situation, as indicated in (11) with the operator $\sup_{\boldsymbol{\Delta} \in \mathcal{R}}$. From [23], we know that $\sup_{\mathbf{y} \in \mathcal{A}} f(\mathbf{x}, \mathbf{y})$ is a convex function in $\mathbf{x}$ if $f(\mathbf{x}, \mathbf{y})$ is also convex in $\mathbf{x}$ for all $\mathbf{y}$, unaffected by the shape of $\mathcal{A}$. This implies that the restrictions in (11) are convex in $\mathbf{P}$ for every possible shape and size of the uncertainty region $\mathcal{R}$ because they are defined as the supremum of a linear (and thus convex) function of $\mathbf{P}$.

In the following section, we particularize the convex problem in (11) for a number of uncertainty regions that model practical situations. The cases of spherical or elliptical uncertainty regions have already been considered in [14], [15] when dealing with robust designs. We try to do a generalization effort to include other different uncertainty regions. In addition, for each uncertainty region, a ready-to-program particularization of the general problem in (11) is given.

IV. Uncertainty Regions

The definition of the uncertainty region $\mathcal{R}$ should take into account the quality of the channel estimate and the imperfections in the estimation process that generate the error

[1]The expression for esinr_i is not a true SINR because in the numerator it has the known contribution of the desired signal and in the denominator it has the interferences and noise powers *plus* the unknown contribution of the desired signal.

in such a way that the mathematical optimization problem in (11) is directly related to the physical phenomenon producing the error. In this section, we focus our attention on the particularization of the general problem in (11) for some interesting uncertainty regions derived from error sources such as estimation Gaussian errors, quantization errors, and combinations of them.

We particularize the obtained expressions for the case where $\mathbf{B} = \mathbf{B}_{\mathrm{ZF}} = \widetilde{\mathbf{H}}^H \big(\widetilde{\mathbf{H}}\widetilde{\mathbf{H}}^H\big)^{-1}$. This choice is made for the sake of simplicity, because, in this case, $|\tilde{\mathbf{h}}_i^H \mathbf{b}_i|^2 = 1$ and $\bar{\mathbf{B}}_i^H \tilde{\mathbf{h}}_i = \mathbf{0}$, for all i, and the general robust problem in (11) becomes

$$\begin{aligned} &\underset{\mathbf{P}}{\text{minimize}} && \operatorname{Tr} \mathbf{B}_{\mathrm{ZF}} \mathbf{P} \mathbf{B}_{\mathrm{ZF}}^H, \\ &\text{subject to} && \sup_{\mathbf{\Delta} \in \mathcal{R}} \boldsymbol{\delta}_i^H \mathbf{B}_{\mathrm{ZF}} \mathbf{P} \mathbf{B}_{\mathrm{ZF}}^H \boldsymbol{\delta}_i - \mathsf{mse}_i^0 p_i + \sigma^2 \leq 0, \ \forall i. \end{aligned} \tag{12}$$

In addition to the simplification of the obtained optimization problem, the choice $\mathbf{B} = \mathbf{B}_{\mathrm{ZF}}$ has been proven asymptotically optimal in terms of signal reception quality at high SNRs and it is also widely utilized in the wireless downlink literature, *e.g.*, [24]. However, it is important to recall that the procedures described below are valid no matter what kind of transmission matrix $\mathbf{B}$ is chosen (see further Section IV-D).

A. Estimation Gaussian Errors

In this section, we deal with the case where the corresponding channel vector of each of the users, $\mathbf{h}_i$, is imperfectly estimated and that the estimate $\tilde{\mathbf{h}}_i$ is an erroneous version of the actual channel corrupted with additive Gaussian noise. This model is valid, for example, if each user estimates its own channel and feeds it back to the transmitter through an ideal feedback link with no quantization or if there is a delay between the estimation and the actual channel in fast varying environments as described in [7] where the author relates the power of the Gaussian error with the Doppler frequency and the delay. The model for the channel estimate is

$$\begin{aligned} &\tilde{\mathbf{h}}_i = \mathbf{h}_i + \mathbf{w}_i, \ \forall i, \quad \text{or, compactly,} \\ &\widetilde{\mathbf{H}} = [\tilde{\mathbf{h}}_1, \ldots, \tilde{\mathbf{h}}_{n_U}]^H = \mathbf{H} + \mathbf{W}, \end{aligned} \tag{13}$$

where $\mathbf{w}_i \in \mathbb{C}^{n_T \times 1}$ represents the estimation error, and whose entries are proper i.i.d. complex Gaussian random variables, $\mathbf{w}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_i)$, and where $\mathbf{W}$ is related with $\mathbf{\Delta}$ as $\mathbf{\Delta} = -\mathbf{W}$. The estimation error power is characterized by $\mathbf{C}_i$ and can be different for each user, so that different qualities in the channel estimation per user can be modeled. The usual approach in this case is to define an elliptical uncertainty region for the error committed in the estimation of the channel of each user (for $\mathbf{C}_i \propto \mathbf{I}$ the region becomes spherical). The uncertainty region is thus formulated as follows

$$\mathcal{R} = \mathcal{E}_{\{R_i^2\},\{\mathbf{C}_i\}} \equiv \left\{ \mathbf{\Delta} \in \mathbb{C}^{n_U \times n_T} \mid \boldsymbol{\delta}_i^H \mathbf{C}_i^{-1} \boldsymbol{\delta}_i \leq R_i^2, \ \forall i \right\}, \tag{14}$$

where R_i is related with the noise power $\operatorname{Tr} \mathbf{C}_i$ and also with the probability that the actual channel is inside the uncertainty region, as detailed in [16]. Note that since the estimation error is assumed Gaussian and the uncertainty region is bounded, the error will lie inside the uncertainty region with a certain probability. Otherwise, an outage event can be declared because the QoS can not be guaranteed (see further [16] to see a numerical evaluation of this outage probability).

From the expression in (14), it can be seen that the quality of the channel estimation for i-th user is determined by the radius of the uncertainty region R_i (the bigger the uncertainty in the channel estimation, the bigger the radius). For example, we can model a situation where the users are placed at different distances of the base station because the uncertainty radius, R_i, of a user which is close to the base station may be lower than that of a user which is placed far from it.

Once the uncertainty region has been properly defined as in (14), the optimization problem in (12) can now be solved. Its solution is given in the following proposition.

Proposition 1: The solution to the optimal power allocation in (12) for the case where $\mathbf{\Delta} \in \mathcal{E}_{\{R_i^2\},\{\mathbf{C}_i\}}$, is given by the solution to the following convex optimization problem:

$$\begin{aligned} &\underset{\mathbf{P}}{\text{minimize}} && \operatorname{Tr} \mathbf{B}_{\mathrm{ZF}} \mathbf{P} \mathbf{B}_{\mathrm{ZF}}^H, \\ &\text{subject to} && R_i^2 \lambda_{\max}(\mathbf{C}_i^{1/2} \mathbf{B}_{\mathrm{ZF}} \mathbf{P} \mathbf{B}_{\mathrm{ZF}}^H \mathbf{C}_i^{1/2}) - \\ & && - \mathsf{mse}_i^0 p_i + \sigma^2 \leq 0, \ \forall i, \end{aligned} \tag{15}$$

which is solved numerically in a very efficient manner following the methods described in [23].

Proof 1: See Appendix I-A.

For the particular case where all the correlation matrices $\mathbf{C}_i$ are equal for all i, $\mathbf{C}_i = \mathbf{C}$, a quasi-closed form solution can be obtained, which is described in the following corollary.

Corollary 1: Let us define the diagonal matrices $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$, with $[\mathbf{\Gamma}]_{ii} = R_i^2/\mathsf{mse}_i^0$ and $[\mathbf{\Sigma}]_{ii} = \sigma^2/\mathsf{mse}_i^0$. Then it follows that the convex problem in (12) for the case $\mathcal{R} = \mathcal{E}_{\{R_i^2\},\{\mathbf{C}\}}$ has a feasible solution if, and only if, $\lambda_{\max}(\mathbf{C}^{1/2} \mathbf{B}_{\mathrm{ZF}} \mathbf{\Gamma} \mathbf{B}_{\mathrm{ZF}}^H \mathbf{C}^{1/2}) < 1$ and its solution, $\mathbf{P}^\star$, is

$$p_i^\star = \frac{R_i^2 \mu + \sigma^2}{\mathsf{mse}_i^0} = \mathsf{esinr}_i^0 \big(R_i^2 \mu + \sigma^2\big), \quad \forall i \in \{1, \ldots, n_U\}, \tag{16}$$

where μ is the unique solution to the fixed point equation $\lambda_{\max}(\mathbf{C}^{1/2} \mathbf{B}_{\mathrm{ZF}} (\mathbf{\Gamma}\mu + \mathbf{\Sigma}) \mathbf{B}_{\mathrm{ZF}}^H \mathbf{C}^{1/2}) = \mu$, which can be very efficiently solved, utilizing, *e.g.*, the Newton method, applying the expression for the differential of $\lambda_{\max}(\mathbf{X})$ in [25, p. 161].

Proof 2: See Appendix I-B

The solution for the optimal power allocation in (16) can be particularized for the case where there is no error in the channel estimate and, consequently, $R_i = 0$, $\forall i$. In this case, $p_{i,\mathrm{perf}}^\star = \mathsf{esinr}_i^0 \sigma^2$. This allows us to interpret that, so that the QoS are guaranteed in the imperfect CSI case, there is an increase of $\mathsf{esinr}_i^0 R_i^2 \mu$ in the power allocated to i-th symbol with respect to the perfect CSI case, which is the minimum price to pay to obtain a robust design for the specific choice of transmitter architecture considered in this work and $\mathbf{B} = \mathbf{B}_{\mathrm{ZF}}$.

B. Quantization Errors

In the previous subsection, we have considered the case where the estimation error is Gaussian. In this section we deal with the case where the available CSI, $\widetilde{\mathbf{H}}$, is a quantized version of the actual channel $\mathbf{H}$. This would correspond to the practical case where each receiver quantizes a perfect

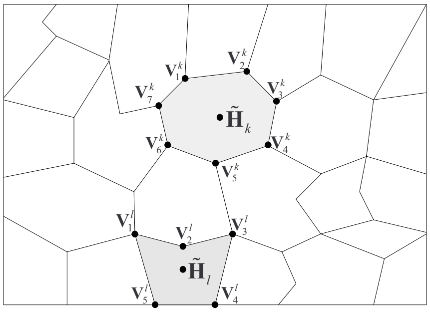

Fig. 3. Graphical representation of the uncertainty regions that arise when the available channel state information is a quantized version of the actual channel. Our formulation allows that each quantization region may have a different shape and that each region may be defined by a different number of vertices, as well as the fact that the quantization regions are not necessarily convex.

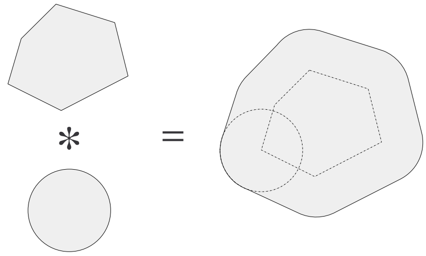

Fig. 4. Graphical representation of the uncertainty region that arises when two different sources of errors come into consideration. The left side of the picture describes the uncertainty regions associated with two sources of errors, *e.g.*, quantization effects (up) and estimation Gaussian errors (down). The resulting uncertainty region (right) is the "convolution" of the two left regions as indicated by the dashed lines.

estimate of its own channel and then feeds back this quantized information to the transmitter through a digital feedback link.

The quantization procedure that we deal with in this section is described in the following. We consider the channel matrices space with K points, $\{\mathbf{H}_k\}$. Each one of these points $\mathbf{H}_k$ is the representative of the region $\mathcal{R}_k$ surrounding it (see Fig. 3). Each region $\mathcal{R}_k$ is a bounded polytope (composed of its boundary and its interior and not necessarily convex) with M_k vertices given by the set $V_k = \{\mathbf{V}_1^k, \mathbf{V}_2^k, \ldots, \mathbf{V}_{M_k}^k\} \subset \mathbb{C}^{n_U \times n_T}$, where the rows of $\mathbf{V}_m^k$ are defined as $\mathbf{V}_m^k = [\mathbf{v}_{m,1}^k, \ldots, \mathbf{v}_{m,n_U}^k]^H$.

The transmitter is informed with the index k of the region where the actual channel belongs to and then the estimate of the channel becomes $\widetilde{\mathbf{H}} = \mathbf{H}_k$ and, consequently, the quantization uncertainty region, $\mathcal{Q}_{V_k}$, becomes the polytope around $\mathbf{H}_k$, $\mathcal{Q}_{V_k} \equiv \mathcal{R}_k$. For the sake of notation, we drop the index k w.l.o.g., and present the characterization of the solution of the problem in (12) in the next proposition.

Proposition 2: Consider the case where $\mathbf{\Delta} \in \mathcal{Q}_V$, then it follows that the convex problem in (12) can be rewritten as the following linear program:

$$\begin{aligned}
&\underset{\mathbf{P}}{\text{minimize}} \quad \operatorname{Tr} \mathbf{B}_{\text{ZF}} \mathbf{P} \mathbf{B}_{\text{ZF}}^H, \\
&\text{subject to} \quad \mathbf{v}_{m,i}^H \mathbf{B}_{\text{ZF}} \mathbf{P} \mathbf{B}_{\text{ZF}}^H \mathbf{v}_{m,i} - \mathsf{mse}_i^0 p_i + \sigma^2 \le 0, \\
&\qquad\qquad \forall i \in \{1, \ldots, n_U\},\ \forall m \in \{1, \ldots, M\}.
\end{aligned} \tag{17}$$

Proof 3: See Appendix I-C.

This general model comprises, as a particular case, the situation where each user uniformly quantizes its corresponding channel coefficients with a different quantization step.

C. Combination of Regions

In realistic setups, the error in the channel matrix may come from more than one source. An example of this situation is the case where the available channel $\widetilde{\mathbf{H}}$ results from a quantization of a corrupted version of the actual channel (see [16]), which corresponds, *e.g.*, to scenarios where each user imperfectly estimates and then quantizes its own channel and then feeds back this quantized channel version to the transmitter.

In this case, the estimation error matrix $\mathbf{\Delta}$ can be considered to be the sum of two terms, $\mathbf{\Delta} = \mathbf{S} + \mathbf{Q}$, the first one takes into account the contribution due to the Gaussian noise, thus $\mathbf{S} \in \mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}$, and the second one models the effects that the channel estimate comes from a quantization process, which implies that $\mathbf{Q} \in \mathcal{Q}_V$. In this case, the shape of the uncertainty region is the hyper-convolution of the two considered regions as we have depicted in Fig. 4, and the solution characterizing the optimal power allocation is given next.

Proposition 3: Consider the case $\mathbf{\Delta} = \mathbf{S} + \mathbf{Q}$, where $\mathbf{S} \in \mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}$ and $\mathbf{Q} \in \mathcal{Q}_V$. Then it follows that the optimization problem in (12) is equivalent to solving the convex program

$$\begin{aligned}
&\underset{\mathbf{P}}{\text{minimize}} \ \operatorname{Tr} \mathbf{B}_{\text{ZF}} \mathbf{P} \mathbf{B}_{\text{ZF}}^H, \\
&\text{subject to} \ \big(\mathbf{s}_{m,i}^\star(\mathbf{P}) + \mathbf{v}_{m,i}\big)^H \mathbf{B}_{\text{ZF}} \mathbf{P} \mathbf{B}_{\text{ZF}}^H \big(\mathbf{s}_{m,i}^\star(\mathbf{P}) + \mathbf{v}_{m,i}\big) - \\
&\qquad\qquad - \mathsf{mse}_i^0 p_i + \sigma^2 \le 0, \quad \forall m, i,
\end{aligned} \tag{18}$$

where $\mathbf{s}_{m,i}^\star(\mathbf{P})$ in (18) depends on $\mathbf{P}$ and it is the solution to the optimization problem described in Appendix II with $\mathbf{A} = \mathbf{B}_{\text{ZF}} \mathbf{P} \mathbf{B}_{\text{ZF}}^H$, $\tilde{\mathbf{y}} = \mathbf{v}_{m,i}$, and $b = R_i^2$.

Proof 4: See Appendix I-D.

Since no closed-form expression for the solution to the convex optimization problem in (18) is apparently available, iterative algorithms are needed to obtain a numerical solution. In this case, at each iteration of the algorithm, the restrictions in (18) have to be numerically evaluated and, consequently, $\mathbf{s}_{m,i}^\star(\mathbf{P})$ has to be computed as a function of the value of $\mathbf{P}$ in the current iteration, as described in Appendix II.

D. Example of Extension to other Types of Transmitters

To illustrate the generality of the methods presented in this work, in this section we give the convex optimization problem whose solution gives the robust power allocation for the general case where we do not impose a particular structure to the transmitter matrix $\mathbf{B}$. As for the definition of the uncertainty region, we consider the very generic case where the estimation error $\mathbf{\Delta}$ is modeled as $\mathbf{\Delta} = \mathbf{E} + \mathbf{Q}$ where $\mathbf{E} \in \mathcal{E}_{\{R_i^2\},\{\mathbf{C}_i\}}$ takes into account the imperfections in the available channel due to colored Gaussian estimation errors and $\mathbf{Q} \in \mathcal{Q}_V$ models the effects of the quantization of the channel estimate. With these assumptions, the general problem in (11) becomes the following convex program

$$\begin{aligned} \underset{\mathbf{P}}{\text{minimize}} \quad & \operatorname{Tr} \mathbf{B}\mathbf{P}\mathbf{B}^H, \\ \text{subject to} \quad & \mathbf{d}_{i,m}^H(\mathbf{P})\mathbf{P}\mathbf{d}_{i,m}(\mathbf{P}) - \\ & - \mathsf{mse}_i^0 \left|\tilde{\mathbf{h}}_i^H \mathbf{b}_i\right|^2 p_i + \sigma^2 \leq 0,\ \forall i, m, \end{aligned} \tag{19}$$

where $\mathbf{d}_{i,m}(\mathbf{P}) = \bar{\mathbf{B}}_i^H \tilde{\mathbf{h}}_i + \mathbf{B}^H(\mathbf{e}_i^\star(\mathbf{P}) + \mathbf{v}_{m,i})$ and where $\mathbf{e}_i^\star(\mathbf{P})$ is the solution to the optimization problem described in Appendix II with $\mathbf{A} = \mathbf{C}_i^{1/2}\mathbf{B}\mathbf{P}\mathbf{B}^H\mathbf{C}_i^{1/2}$, $\tilde{\mathbf{y}} = \mathbf{C}_i^{-1/2}\left(\mathbf{v}_{m,i} + \mathbf{B}\big(\mathbf{B}^H\mathbf{B}\big)^{-1}\bar{\mathbf{B}}_i^H\tilde{\mathbf{h}}_i\right)$, and $b = R_i^2$. This statement is left without proof because it is very similar to that of Proposition 3. Note that the convex optimization problem in (19) admits, as particular cases, the convex problems obtained previously.

V. Practical Issues

When numerically solving convex optimization problems such as (15), (17), (18), and (19) where no closed form solution is available, there are two main considerations that have to be taken into account. First of all, it is important to study the feasibility of the problem because if the feasible set is empty, then, no solution exists. Secondly, if the feasible region is non-empty, a feasible initial value of the optimization variable, $\mathbf{P}^{(0)}$, has to be provided to the iterative numerical optimization algorithm. Both issues are discussed in the following.

A. Feasibility

First of all, note that the restrictions in the problems (15), (17), (18), and (19) can be equivalently expressed as a list of restrictions indexed by i and/or m. For the sake of notation we now define q as the index of the list, which implies that the restrictions in (15), (17) and (18) are of the general form $r_q(\mathbf{P}) + \sigma^2 \leq 0$, $\forall q$, where $r_q(\mathbf{P})$ is a convex function in $\mathbf{P}$ and homogenous of degree 1, because the function $r_q(\mathbf{P})$ is defined as the supremum of a set of linear functions of $\mathbf{P}$, *i.e.*, it fulfills that $r_q(\alpha\mathbf{P}) = \alpha r_q(\mathbf{P})$. Clearly, from all this set of restrictions, if $r_q(\mathbf{P}) + \sigma^2 \leq 0$ is met for the maximum w.r.t. q then is met for all q. Thus, we define $r(\mathbf{P}) = \max_q r_q(\mathbf{P})$, which is also homogeneous of degree 1. An equivalent restriction to $r_q(\mathbf{P}) + \sigma^2 \leq 0$ for the feasibility problem is then $r(\mathbf{P}) + \sigma^2 \leq 0$. Now, if $\mathbf{P}$ is feasible, then $\alpha\mathbf{P}$ with $\alpha > 1$ is also feasible. In particular, the limit case for $\alpha \to \infty$ is also feasible, and, in this limit case, the noise variance σ^2 does not have any influence in $r(\mathbf{P}) + \sigma^2 \leq 0$, and therefore the feasibility problem is equivalent to proving the existence of a $\mathbf{P}$ matrix such that $r(\mathbf{P}) < 0$, which can be further simplified to $r(\bar{\mathbf{P}}) < 0$, with $\operatorname{Tr}\bar{\mathbf{P}} = 1$, where we have defined $\bar{\mathbf{P}} = \mathbf{P}/\operatorname{Tr}\mathbf{P}$ and the homogeneity property $r(\alpha\mathbf{P}) = \alpha r(\mathbf{P})$ has been utilized. Since we are only interested in proving the existence of a matrix that fulfills $r(\bar{\mathbf{P}}) < 0$ we can restrict our attention to the matrix that is most *likely* to fulfill it, *i.e.*, the matrix that minimizes the term $r(\bar{\mathbf{P}})$, which is the solution to

$$\begin{aligned} \underset{\bar{\mathbf{P}}}{\text{minimize}} \quad & r(\bar{\mathbf{P}}), \\ \text{subject to} \quad & \operatorname{Tr}\bar{\mathbf{P}} = 1, \quad \bar{p}_i \geq 0,\ \forall i. \end{aligned} \tag{20}$$

Since $r(\bar{\mathbf{P}})$ is defined as the maximum of a finite set of convex functions it is also convex, and consequently the optimization problem in (20) is also convex and its solution always exists, can be efficiently calculated, and it is denoted by $\bar{\mathbf{P}}^\star$.

Once we have obtained this solution, it only remains to check whether $r(\bar{\mathbf{P}}^\star) < 0$ and the problem is feasible or $r(\bar{\mathbf{P}}^\star) \geq 0$, which means that the problem is infeasible because $r(\bar{\mathbf{P}}) \geq r(\bar{\mathbf{P}}^\star) \geq 0$. If the problem is infeasible it means that there exists no power allocation such that all the QoS constraints can be fulfilled. In this case, an outage event can be declared, or, alternatively, some of the QoS constraints could be relaxed.

B. Starting Point

If the problem is feasible, *i.e.*, if $r(\bar{\mathbf{P}}^\star) < 0$, then a solution to the considered original problem (15), (17), (18) or (19) exists, and efficient numerical algorithms can be utilized to calculate it. As the numerical algorithms need a starting feasible point, $\mathbf{P}^{(0)}$, to begin the iterative procedure, we propose a starting point of the form $\mathbf{P}^{(0)} = \beta\bar{\mathbf{P}}^\star$, where β is calculated so that $\mathbf{P}^{(0)}$ is feasible, *i.e.*, $\mathbf{P}^{(0)}$ has to fulfill $r(\mathbf{P}^{(0)}) + \sigma^2 \leq 0$:

$$\begin{aligned} & r(\mathbf{P}^{(0)}) + \sigma^2 \leq 0 \Rightarrow r(\beta\bar{\mathbf{P}}^\star) + \sigma^2 \leq 0 \Rightarrow \\ & \qquad \Rightarrow \beta r(\bar{\mathbf{P}}^\star) + \sigma^2 \leq 0 \Rightarrow \beta \geq \frac{-\sigma^2}{r(\bar{\mathbf{P}}^\star)}. \end{aligned} \tag{21}$$

Note that the feasibility of the problem ($r(\bar{\mathbf{P}}^\star) < 0$) is necessary to guarantee that β exists and is positive, and consequently $\mathbf{P}^{(0)}$ is positive semi-definite as expected.

VI. Numerical Examples

In the following, some numerical examples are provided in order to give more insight into the benefits of the proposed robust design for the power allocation.

In Fig. 5, we have considered a two user scenario where the uncertainty matrix belongs to a spherical region as discussed in Section IV-A. We have plotted the feasibility region (*i.e.*, the set of powers p_1 and p_2 for which the QoS constraints are fulfilled $\forall\mathbf{\Delta} \in \mathcal{R}$) for different values of the uncertainty radius, $R_i = R$, $\forall i$. Note that as the uncertainty radius increases the region becomes smaller. If we continued to increase the radius (*i.e.*, the uncertainty) there would be a point where the feasibility region becomes void and, thus, the optimization problem is infeasible.

Within the same scenario, in Fig. 6, we have fixed the value of the uncertainty radius for user 2, R_2. In the upper plot, we have drawn the feasibility region (*i.e.*, the set of esinr_i^0 such that the problem is feasible) for different values of the uncertainty radius of user 1, R_1. The feasibility region corresponds to the area below each one of the curves. In the lower plot, we have represented the total transmitted power along the red dotted line. Note that, as we approach the limit of the feasibility region, the necessary transmitted power to guarantee the QoS constraints becomes arbitrarily large.

In addition we have conducted some numerical evaluations to show the goodness of our proposed robust method. Note that, although in [4], [21], [22] a robust multiuser transmitter

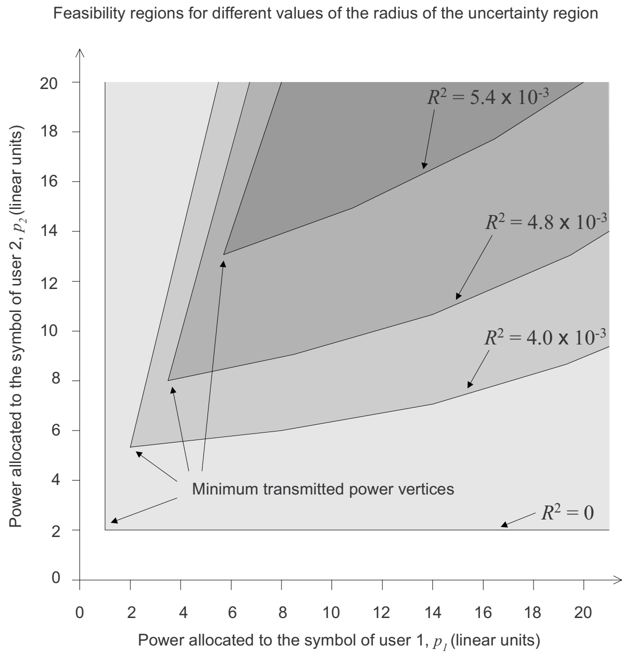

Fig. 5. Feasibility region as a function of the power allocated to the symbol of each user for different values of the radius of the uncertainty region. The lower-left corner of each region corresponds to the feasible point (*i.e.* the QoS constraints are met) such that the transmitted power is the lowest, *i.e.* the robust power allocation.

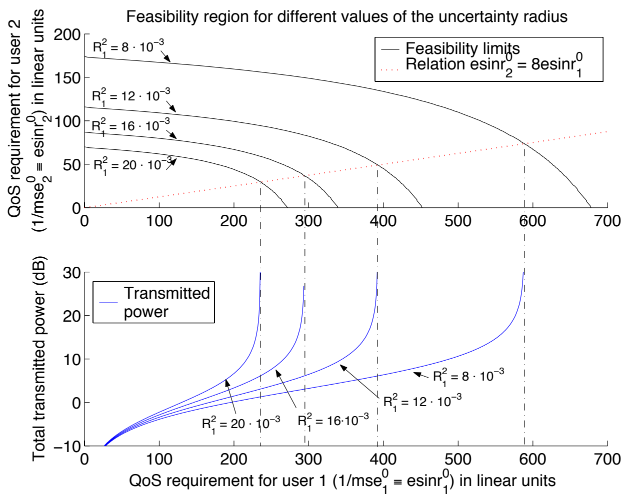

Fig. 6. The upper plot depicts the feasibility region as a function of the QoS requirement for each user for different values of the radius of the uncertainty region. The lower plot represents the total transmitted power following the dotted red line.

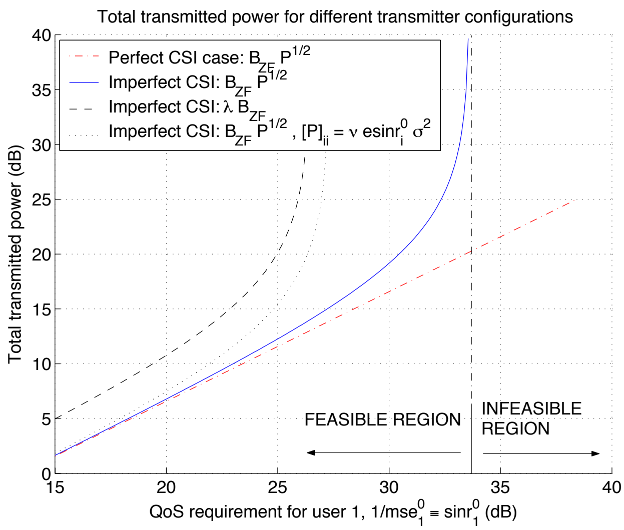

Fig. 7. Total transmitted power versus QoS requirement for user 1, esinr_1^0. In this scenario we considered eight users, $n_U = 8$.

design is also addressed, a fair comparison of these scheme with ours is not possible because the uncertainty model is different (in these works the uncertainty is on the channel *covariance* matrix and not in the channel itself). Instead, we have evaluated the performance of different linear transmitter configurations. For the fixed matrix $\mathbf{B}$ we have selected the matrices $\mathbf{B}_{\mathrm{ZF}}$, $\mathbf{B}_{\mathrm{WF}} = (\mathbf{H}^H\mathbf{H}+\alpha\mathbf{I})^{-1}\mathbf{H}^H$ as in [3], [26], and $\mathbf{B} = \mathbf{I}$. For the power allocation matrix, in addition to the general form $\mathbf{P}^{1/2}$, we have also considered the suboptimal approaches of forcing $\mathbf{P}^{1/2} = \lambda\mathbf{I}$, and $[\mathbf{P}]_{ii} = \nu \mathsf{esinr}_i^0\sigma^2$, ($\lambda$ and ν are such that all QoS constraints are fulfilled).

In Fig. 7 we have fixed $\mathbf{B} = \mathbf{B}_{\mathrm{ZF}}$ and we have considered an eight user scenario with a spherical uncertainty region (Section IV-A). For the perfect and imperfect CSI cases, the necessary total transmitted power is plotted as a function of the QoS requirement for user 1, esinr_1^0, while the QoS requirements for the other users are fixed. As expected, the robust solution $\mathbf{B}_{\mathrm{ZF}}\mathbf{P}^{1/2}$ yields the minimum necessary transmitted power to guarantee the QoS constraints. Note also that, with the same QoS constraints, there is an increase in the minimum necessary transmitted power for the case of imperfect CSI with respect to the perfect CSI case.

A similar numerical evaluation has been conducted in Fig. 8, for a four user scenario. In this case, we have considered different configurations for the transmitter matrix $\mathbf{B}$. While for low QoS requirements, the best configuration is $\mathbf{B}_{\mathrm{WF}}\mathbf{P}^{1/2}$, for high QoS requirements the best option is $\mathbf{B}_{\mathrm{ZF}}\mathbf{P}^{1/2}$. Note how always the simplified structure $\mathbf{P}^{1/2} = \lambda\mathbf{I}$ yields a worse performance than allowing all the degrees of freedom in $\mathbf{P}$, supporting our approach.

VII. Conclusions

In this paper, a multiantenna downlink multiuser system has been considered, where the power allocation among the data streams of different users has been designed in a robust way against uncertainties and errors in the available CSI. The robustness has been formulated under a worst-case framework, where the objective has been the minimization of the total transmitted power while still guaranteeing a minimum QoS per user for any possible channel realization within the uncertainty region around the available CSI. The robust power allocation design has been solved, either numerically or in quasi-closed form, using the tools provided by convex optimization theory. Besides, some practical aspects related to the feasibility of the problem and the starting point in case of using numerical methods have also been studied.

By means of numerical results, it has been proved that the proposed design of the transmitter $\mathbf{BP}^{1/2}$ improves the

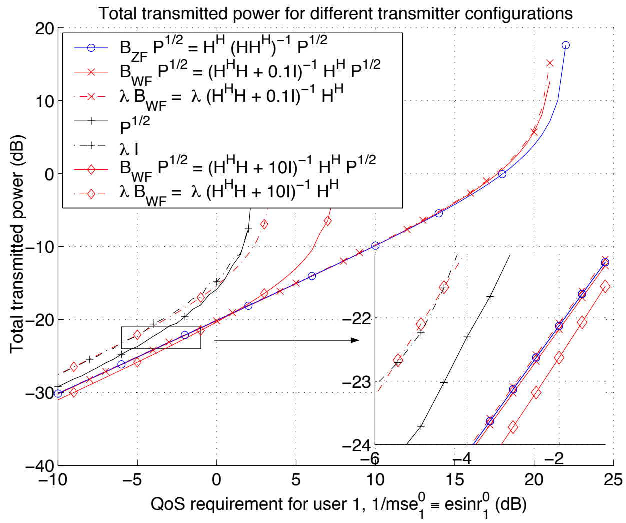

Fig. 8. Total transmitted power versus QoS requirement for user 1, esinr_1^0. The scenario considered in this numerical example has four users and four transmit antennas, $n_U = n_T = 4$.

performance achieved by other power allocation policies, such as scaled versions like $\lambda\mathbf{B}$, for the cases where $\mathbf{B} = \mathbf{B}_{\text{ZF}}$, $\mathbf{B} = \mathbf{B}_{\text{WF}}$, and $\mathbf{B} = \mathbf{I}$. We have shown that our robust technique needs less power than other solutions while still guaranteeing the same QoS.

Further interesting work may include the consideration of different channel estimates available at both communication ends and the joint optimization of $\mathbf{B}$ and $\mathbf{P}^{1/2}$. However, both extensions appear to be difficult.

Appendix I

A. Proof of Proposition 1

We first particularize the general convex problem in (12) for $\mathcal{R} = \mathcal{E}_{\{R_i^2\},\{\mathbf{C}_i\}}$ as in (14). Since in the definition of the uncertainty region, there are independent restrictions for each row $\boldsymbol{\delta}_i^H$ of the uncertainty matrix $\boldsymbol{\Delta}$, the supremum in the restriction of the problem in (11) becomes

$$\sup_{\boldsymbol{\Delta}\in\mathcal{E}_{\{R_i^2\},\{\mathbf{C}_i\}}} \boldsymbol{\delta}_i^H \mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H \boldsymbol{\delta}_i \equiv \left[\begin{array}{l} \sup\limits_{\boldsymbol{\delta}_i} \boldsymbol{\delta}_i^H \mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H \boldsymbol{\delta}_i \\ \text{s.t.}\ \ \boldsymbol{\delta}_i^H \mathbf{C}_i^{-1} \boldsymbol{\delta}_i \le R_i^2 \end{array}\right]. \quad (22)$$

Last problem can be solved by performing the change $\tilde{\boldsymbol{\delta}}_i = \mathbf{C}_i^{-1/2}\boldsymbol{\delta}_i$. It is now well known that the solution is given by $\tilde{\boldsymbol{\delta}}_i$ being proportional to the eigenvector associated with the maximum eigenvalue of $\mathbf{C}_i^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H\mathbf{C}_i^{1/2}$ and such that $\tilde{\boldsymbol{\delta}}_i^H \tilde{\boldsymbol{\delta}}_i = R_i^2$. The problem in (11) becomes

$$\begin{aligned} &\underset{\mathbf{P}}{\text{minimize}} && \operatorname{Tr}\mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H, \\ &\text{subject to} && R_i^2\lambda_{\max}(\mathbf{C}_i^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H\mathbf{C}_i^{1/2}) - \\ &&& - \text{mse}_i^0 p_i + \sigma^2 \le 0,\ \forall i. \end{aligned} \quad (23)$$

B. Proof of Corollary 1

Let $\mathbf{C}_i = \mathbf{C}$, $\forall i$. It is straightforward to prove that the solution $\mathbf{P}^\star$ to the problem (15) fulfills the restrictions with equality. Then, defining $\mu = \lambda_{\max}(\mathbf{C}^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{P}^\star\mathbf{B}_{\text{ZF}}^H\mathbf{C}^{1/2}) \ge 0$, we obtain that the optimal power allocation must fulfill $p_i^\star = (R_i^2\mu + \sigma^2)/\text{mse}_i^0$, where μ has to be determined. First of all, we utilize the definitions of the diagonal matrices $[\boldsymbol{\Gamma}]_{ii} \equiv R_i^2/\text{mse}_i^0$ and $[\boldsymbol{\Sigma}]_{ii} \equiv \sigma^2/\text{mse}_i^0$ to express $\mathbf{P}^\star = \boldsymbol{\Gamma}\mu + \boldsymbol{\Sigma}$. Then, from the restriction in (15) for the optimal power allocation $\mathbf{P}^\star$ we obtain the following equation for the μ parameter

$$\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}}) = \mu, \quad (24)$$

$$\text{with } \begin{cases} \boldsymbol{\Gamma}_{\mathbf{B}} \equiv \mathbf{C}^{1/2}\mathbf{B}_{\text{ZF}}\boldsymbol{\Gamma}\mathbf{B}_{\text{ZF}}^H\mathbf{C}^{1/2} \\ \boldsymbol{\Sigma}_{\mathbf{B}} \equiv \mathbf{C}^{1/2}\mathbf{B}_{\text{ZF}}\boldsymbol{\Sigma}\mathbf{B}_{\text{ZF}}^H\mathbf{C}^{1/2} \end{cases}, \quad (25)$$

where both sides in (24) are convex functions of μ. We now find under which conditions, equation (24) has a solution (and how many). Since $\boldsymbol{\Gamma}_{\mathbf{B}}$ and $\boldsymbol{\Sigma}_{\mathbf{B}}$ are positive definite matrices, then

$$\begin{aligned} &\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}}) \le \lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}})\mu + \lambda_{\max}(\boldsymbol{\Sigma}_{\mathbf{B}}), \\ &\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}}) > \lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}})\mu. \end{aligned} \quad (26)$$

Last inequality clearly implies that if $\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}) \ge 1$ then (24) has no solution because no intersection in the fixed point equation (24) is possible, as for all $\mu \ge 0$, then $\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}}) > \lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}})\mu \ge \mu$. On the contrary, if $\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}) < 1$, then there exists $\mu \ge 0$ such that $\mu > \lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}})\mu + \lambda_{\max}(\boldsymbol{\Sigma}_{\mathbf{B}}) \ge \lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}})$, and for $\mu = 0$ then $\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}}) > \mu$. From continuity it can be deduced that (24) must have a solution. From the fact that $\lambda_{\max}(\boldsymbol{\Gamma}_{\mathbf{B}}\mu + \boldsymbol{\Sigma}_{\mathbf{B}})$ is a monotonically increasing and convex function only one solution can exist.

C. Proof of Proposition 2

In this case, it can be seen that the uncertainty region $\mathcal{Q}_V$ is the polytope whose vertices are given by the set $V = \{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_M\} \subset \mathbb{C}^{n_U \times n_T}$. The convex hull of the set of points V, denoted by $\operatorname{conv} V$, is the minimal convex set containing V, [23], which clearly implies that the polytope $\mathcal{Q}_V$ is inside the convex hull of V, *i.e.*, $\mathcal{Q}_V \subseteq \operatorname{conv} V$. The convex hull of V is [23]

$$\operatorname{conv} V = \left\{ \sum_{m=1}^M \theta_m \mathbf{V}_m \;\middle|\; \sum_{m=1}^M \theta_m = 1,\ \theta_m \ge 0,\ \forall m \right\}. \quad (27)$$

We define $f(\boldsymbol{\Delta}) \equiv \boldsymbol{\delta}_i^H \mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H\boldsymbol{\delta}_i$, which is a convex function in $\boldsymbol{\Delta}$ (and in $\boldsymbol{\delta}_i$). The supremum in the restriction in (12) particularizes to $\sup_{\boldsymbol{\Delta}\in\mathcal{Q}_V} f(\boldsymbol{\Delta})$ which is bounded by

$$\sup_{\boldsymbol{\Delta}\in\mathcal{Q}_V} f(\boldsymbol{\Delta}) \le \sup_{\boldsymbol{\Delta}\in\operatorname{conv} V} f(\boldsymbol{\Delta}), \quad (28)$$

where $\mathcal{Q}_V \subseteq \operatorname{conv} V$ has been used. From (27), for any $\boldsymbol{\Delta} \in \operatorname{conv} V$, it exists a set $\{\theta_1, \ldots, \theta_M : \sum_{m=1}^M \theta_m = 1, \theta_m \ge 0, \forall m\}$ such that

$$\begin{aligned} \boldsymbol{\Delta} \in \operatorname{conv} V \Rightarrow f(\boldsymbol{\Delta}) = f\left(\sum\nolimits_{m=1}^M \theta_m \mathbf{V}_m\right) \le \\ \le \sum_{m=1}^M \theta_m f(\mathbf{V}_m) \le \max_m f(\mathbf{V}_m). \end{aligned} \quad (29)$$

From last equation we can deduce that $\sup_{\boldsymbol{\Delta}\in\operatorname{conv} V} f(\boldsymbol{\Delta}) \le \max_m f(\mathbf{V}_m)$. Moreover, since $\mathbf{V}_m \in \mathcal{Q}_V$, this implies that

$\sup_{\boldsymbol{\Delta}\in\mathcal{Q}_V} f(\boldsymbol{\Delta}) = \max_m f(\mathbf{V}_m)$, which means that the supremum of $f(\boldsymbol{\Delta})$, with $\boldsymbol{\Delta}\in\mathcal{Q}_V$, has to be necessarily placed in one of the vertices of the polytope (a priori we do not know which one, though). Consequently, the supremum operation $\sup_{\boldsymbol{\Delta}\in\mathcal{Q}_V} f(\boldsymbol{\Delta})$ can be substituted by a simple list of the function $f(\boldsymbol{\Delta})$ evaluated at all the different vertices since necessarily one of them has to be the supremum, $f(\mathbf{V}_m)=\mathbf{v}_{m,i}^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H\mathbf{v}_{m,i}$, where $\mathbf{v}_{m,i}^H$ is the i-th row of the vertex $\mathbf{V}_m$. The equivalent optimization problem is thus:

$$\begin{aligned}\underset{\mathbf{P}}{\text{minimize}}\quad & \operatorname{Tr}\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H,\\ \text{subject to}\quad & \mathbf{v}_{m,i}^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H\mathbf{v}_{m,i}-\mathsf{mse}_i^0 p_i+\sigma^2\le 0,\ \forall i,m.\end{aligned}$$

D. Proof of Proposition 3

In this section we have to consider the case where $\boldsymbol{\Delta}=\mathbf{S}+\mathbf{Q}$, with $\mathbf{S}\in\mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}$ and $\mathbf{Q}\in\mathcal{Q}_V$. Consequently, in this case we have $\boldsymbol{\delta}_i=\mathbf{s}_i+\mathbf{q}_i$ and the particularization of the restriction of the problem in (12) for this uncertainty region becomes [23]

$$\begin{aligned}&\sup_{\substack{\mathbf{S}\in\mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}\\ \mathbf{Q}\in\mathcal{Q}_V}}\ (\mathbf{s}_i+\mathbf{q}_i)^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{q}_i)\equiv\\ &\equiv\sup_{\mathbf{S}\in\mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}}\ \sup_{\mathbf{Q}\in\mathcal{Q}_V}\ (\mathbf{s}_i+\mathbf{q}_i)^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{q}_i).\end{aligned}\tag{30}$$

Now, we note that $(\mathbf{s}_i+\mathbf{q}_i)^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{q}_i)$ is a convex function of $\mathbf{q}_i$ and, similarly as we have done in the previous section, the solution of the inner maximization is given by the function $(\mathbf{s}_i+\mathbf{q}_i)^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{q}_i)$ evaluated at one of the vertices that define the convex hull $\{\mathbf{V}_m\}$. From the original problem in (30), we obtain a set of maximization problems indexed by the variable m that represent the vertices index, as

$$\begin{aligned}&\sup_{\mathbf{S}\in\mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}}\ (\mathbf{s}_i+\mathbf{v}_{m,i})^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{v}_{m,i})\equiv\\ &\qquad\equiv\sup_{\mathbf{s}_i^H\mathbf{s}_i\le R_i^2}\ (\mathbf{s}_i+\mathbf{v}_{m,i})^H\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H(\mathbf{s}_i+\mathbf{v}_{m,i}),\end{aligned}\tag{31}$$

where we have utilized the definition of the hyper-spherical uncertainty region $\mathcal{E}_{\{R_i^2\},\{\mathbf{I}\}}$. The problem expressed in (31) is solved in Appendix II, with $\tilde{\mathbf{x}}=\mathbf{s}_i$, $\tilde{\mathbf{y}}=\mathbf{v}_{m,i}$, $\mathbf{A}=\mathbf{B}_{\mathrm{ZF}}\mathbf{P}\mathbf{B}_{\mathrm{ZF}}^H$, and $b=R_i^2$ and its solution is denoted by $\mathbf{s}_{i,m}^\star(\mathbf{P})$.

APPENDIX II
MAXIMIZATION OF A GENERAL QUADRATIC FORM WITH A NORM CONSTRAINT

Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be vectors in the field $\mathbb{C}^{n_T\times 1}$ and let $\mathbf{A}\in\mathbb{C}^{n_T\times n_T}$ be a positive semi-definite matrix with $n_U\le n_T$ non-zero eigenvalues. We want to solve

$$\begin{aligned}\underset{\tilde{\mathbf{x}}}{\text{maximize}}\quad & (\tilde{\mathbf{x}}+\tilde{\mathbf{y}})^H\mathbf{A}(\tilde{\mathbf{x}}+\tilde{\mathbf{y}}),\\ \text{subject to}\quad & \tilde{\mathbf{x}}^H\tilde{\mathbf{x}}\le b,\quad (b>0).\end{aligned}\tag{32}$$

Performing the SVD decomposition of the positive semi-definite $\mathbf{A}$ matrix we obtain $\mathbf{A}=\mathbf{U}\boldsymbol{\Omega}\mathbf{U}^H$, with $\boldsymbol{\Omega}\in\mathbb{C}^{n_T\times n_T}$ being a positive semi-definite diagonal matrix, and with $\mathbf{U}\in\mathbb{C}^{n_T\times n_T}$ being unitary. Introducing the changes $\mathbf{x}'=\mathbf{U}^H\tilde{\mathbf{x}}$ and $\mathbf{y}'=\mathbf{U}^H\tilde{\mathbf{y}}$ the problem in (32) becomes

$$\begin{aligned}\underset{\{x_i'\}_{i=1}^{n_T}}{\text{maximize}}\quad & \textstyle\sum_{i=1}^{n_U}|x_i'+y_i'|^2\omega_i,\\ \text{subject to}\quad & \textstyle\sum_{i=1}^{n_T}|x_i'|^2\le b,\end{aligned}\tag{33}$$

where $\omega_i=[\boldsymbol{\Omega}]_{ii}>0$ for $i\in[1,n_U]$, and where the remaining $\omega_i=0$ for $i\in[n_U+1,n_T]$ have been discarded from the summation in the objective function in (33). Then clearly the optimal solution $\mathbf{x}'^\star$ fulfills $x_i'^\star=0$ for $i\in[n_U+1,n_T]$. Noting that the inequality $|x_i'+y_i'|^2\le\left||x_i'|+|y_i'|\right|^2$ becomes an equality if the complex phases of x_i' and y_i' are the same, then we can state that $\angle x_i'^\star=\angle y_i'$.

Once we know the complex phase of the solution, the problem becomes a real optimization problem by defining $x_i=|x_i'|$ and $y_i=|y_i'|$, and then the problem in (33) becomes

$$\begin{aligned}\underset{\{x_i\}_{i=1}^{n_U}}{\text{maximize}}\quad & \textstyle\sum_{i=1}^{n_U}(x_i+y_i)^2\omega_i,\\ \text{subject to}\quad & \textstyle\sum_{i=1}^{n_U}x_i^2\le b,\ \ x_i\ge 0,\ \forall i,\end{aligned}\tag{34}$$

where the set $\{y_i\}$ and b are all real non-negative numbers.

We now partition the terms in the summation in (34) in two groups $\mathcal{I}_1$ and $\mathcal{I}_2$, depending on whether its corresponding y_i is greater than zero ($y_i>0\Leftrightarrow i\in\mathcal{I}_1\subset[1,n_U]$) or equal to zero ($y_i=0\Leftrightarrow i\in\mathcal{I}_2\subset[1,n_U]$), with $\mathcal{I}_1\cup\mathcal{I}_2=[1,n_U]$. Finally, defining $z_i=-(x_i+y_i)^2$, for all i, we obtain a convex problem, equivalent to (34), as

$$\begin{aligned}\underset{\{z_i\}_{i=1}^{n_U}}{\text{minimize}}\quad & -\sum_{i\in\mathcal{I}_1}z_i\omega_i-\sum_{i\in\mathcal{I}_2}z_i\omega_i,\\ \text{subject to}\quad & \sum_{i\in\mathcal{I}_1}\left(\sqrt{z_i}-y_i\right)^2+\sum_{i\in\mathcal{I}_2}z_i-b\le 0,\\ & y_i^2-z_i\le 0,\ \forall i.\end{aligned}\tag{35}$$

Note that we can assume w.l.o.g. that for any two indices $k,l\in\mathcal{I}_2$ then $k\ne l\Rightarrow\omega_k\ne\omega_l$. This can be assumed because in case $\omega_k=\omega_l$ for some different $k,l\in\mathcal{I}_2$, then we can always define a new variable $z_{kl}=z_k+z_l$ such that the equivalent problem has the same structure as the original one and is one dimension smaller. Furthermore, we can also assume w.l.o.g. that $k\in\mathcal{I}_1$, $l\in\mathcal{I}_2\Rightarrow\omega_k\ne\omega_l$. In this case, the proof follows from a primal decomposition [27] of the original problem in (35) with $\omega_l=\omega_k$ for some $k\in\mathcal{I}_1$ and $l\in\mathcal{I}_2$. The primal decomposition is performed by separating the terms with indices k and l in the objective function and by adding the auxiliary variable $c\ge 0$ that allows us to decouple the first restriction in (35) for k and l as it can be seen in (36) at the bottom of the next page. It can be shown that the solution to the inner minimization problem in (36) yields $z_l^\star=0$ for all possible values of c, y_k, and ω_k which implies that in case $\omega_k=\omega_l$ for some $k\in\mathcal{I}_1$ and $l\in\mathcal{I}_2$ the term corresponding to the index l can be eliminated w.l.o.g. From all that has been said, it follows that the multiplicity of ω_i is only possible among indices inside the set $\mathcal{I}_1$ but not inside the set $\mathcal{I}_2$ or between one element of $\mathcal{I}_1$ and one of $\mathcal{I}_2$. Because in the latter cases an equivalent problem is obtained without this multiplicity.

From the KKT conditions of (35), we obtain

$$-\omega_i + \lambda^\star \left(1 - \frac{y_i}{\sqrt{z_i^\star}}\right) - \mu_i^\star = 0,\ \forall i \in \mathcal{I}_1, \quad (37)$$

$$\mu_i^\star \left(y_i^2 - z_i^\star\right) = 0,\ \forall i \in \mathcal{I}_1, \quad (38)$$

$$-\omega_i + \lambda^\star - \mu_i^\star = 0,\ \forall i \in \mathcal{I}_2, \quad (39)$$

$$-\mu_i^\star z_i^\star = 0,\ \forall i \in \mathcal{I}_2, \quad (40)$$

$$\lambda^\star \left(\sum_{i\in\mathcal{I}_1} \left(\sqrt{z_i^\star} - y_i\right)^2 + \sum_{i\in\mathcal{I}_2} z_i^\star - b\right) = 0, \quad (41)$$

$$\lambda^\star \geq 0, \quad (42)$$

$$\mu_i^\star \geq 0,\ \forall i. \quad (43)$$

In the following, we deduce that $\mu_i^\star = 0, \forall i \in \mathcal{I}_1$. Note that from the last restriction in (35), either $z_i^\star = y_i^2$ or $z_i^\star > y_i^2$ must hold. If $z_i^\star > y_i^2$, then from (38) we obtain $\mu_i^\star = 0$. On the contrary, if $z_i^\star = y_i^2$ then (37) becomes $-\omega_i - \mu_i^\star = 0$. The inequalities $\omega_i > 0$ (by definition) and $\mu_i^\star \geq 0$ (from (43)) imply that $-\omega_i - \mu_i^\star < 0$ which is a contradiction; thus, $z_i^\star = y_i^2$ is not a possible solution. Consequently, for $i \in \mathcal{I}_1$ we obtain $z_i^\star > y_i^2$ and $\mu_i^\star = 0$. From (37) with $\mu_i^\star = 0$, we obtain the solution to (35) for $i \in \mathcal{I}_1$ as

$$z_i^\star = \left(\frac{y_i \lambda^\star}{\lambda^\star - \omega_i}\right)^2, \quad \forall i \in \mathcal{I}_1, \quad (44)$$

where $\lambda^\star$ has yet to be determined.

We now focus on the set of indices $\mathcal{I}_2$. Note that it is impossible that more than one optimal $z_{i'}^\star$ be greater than zero with $i' \in \mathcal{I}_2$. If we assume that $z_{i'}^\star > 0$ and $z_{j'}^\star > 0$, from (40) it would imply that $\mu_{i'}^\star = 0$ and $\mu_{j'}^\star = 0$ which in its turn would mean that $\lambda^\star = \omega_{i'}$ and $\lambda^\star = \omega_{j'}$ which is impossible, because $\omega_{i'} \neq \omega_{j'}$, $\forall i', j' \in \mathcal{I}_2$, and $\lambda^\star$ can take only one value. Consequently, at most there exists one index $i' \in \mathcal{I}_2$ such that $z_{i'}^\star > 0$. Obviously, in case this index exists, it corresponds to $i'_{\max} = \arg\max_{i\in\mathcal{I}_2} \omega_i$.

From all said above, only two cases need to be considered:

1) $z_i^\star = 0, \forall i \in \mathcal{I}_2$. Then, the optimal solution is completed with (44), where $\lambda^\star$ is determined from the restriction in (35), similarly as in [28], as the biggest solution to the equation

$$\sum_i \left(\frac{y_i \omega_i}{\lambda^\star - \omega_i}\right)^2 - b = 0. \quad (45)$$

2) There exists an index $i'_{\max} \in \mathcal{I}_2$ such that $z_{i'_{\max}}^\star > 0$. Then, from (40), $\mu_{i'_{\max}}^\star = 0$ and thus $\lambda^\star = \omega_{i'_{\max}}$ as indicated by (39). Plugging this value for $\lambda^\star$ in (44) we obtain the solution

$$z_i^\star = \left(\frac{y_i \omega_{i'_{\max}}}{\omega_{i'_{\max}} - \omega_i}\right)^2, \quad \forall i \in \mathcal{I}_1, \quad (46)$$

where the denominator is always different of zero as $i \in \mathcal{I}_1$ and $i'_{\max} \in \mathcal{I}_2$ which implies that $\omega_i \neq \omega_{i'_{\max}}$. The solution is completed with $z_{i'_{\max}}^\star$, which is determined such that the power constraint is fulfilled with equality:

$$z_{i'_{\max}}^\star = b - \sum_{i\in\mathcal{I}_1} \left(\sqrt{z_i^\star} - y_i\right)^2. \quad (47)$$

To determine which of the two cases is the optimal one, we only need to calculate (46) and then check the sign of $b - \sum_{i\in\mathcal{I}_1} \left(\sqrt{z_i^\star} - y_i\right)^2$. If it is negative then (47) becomes meaningless because $z_{i'}^\star$ has to be positive or zero and thus the solution is given by the first case. Otherwise, the solution is given by the second case. Once we know the solution to the convex problem in (35), $\{z_i^\star\}_{i=1}^{n_U}$, we simply need to construct the solution $\tilde{\mathbf{x}}^\star$ of the original problem in (32) as $\tilde{\mathbf{x}}^\star = \mathbf{U}\mathbf{x}'^\star$, with $x_i'^\star = (\sqrt{z_i^\star} - y_i) \cdot \exp\left(\mathrm{j}\angle(y_i')\right)$ if $i \in [1, n_U]$, and $x_i'^\star = 0$ if $i \in [n_U + 1, n_T]$.

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$$\begin{aligned}
\underset{\{z_i\}_{i=1}^{n_U}\setminus\{z_k,z_l\},c\geq 0}{\text{minimize}} \quad & -\sum_{i\in\mathcal{I}_1\setminus\{k\}} z_i\omega_i - \sum_{i\in\mathcal{I}_2\setminus\{l\}} z_i\omega_i + \begin{cases} \underset{z_k,z_l}{\text{minimize}} & -\omega_k z_k - \omega_k z_l, \\ \text{subject to} & \left(\sqrt{z_k} - y_k\right)^2 + z_l - c \leq 0, \end{cases} \\
\text{subject to} \quad & \sum_{i\in\mathcal{I}_1\setminus\{k\}} \left(\sqrt{z_i} - y_i\right)^2 + \sum_{i\in\mathcal{I}_2\setminus\{l\}} z_i + c - b \leq 0, \\
& y_i^2 - z_i \leq 0,\ \forall i.
\end{aligned} \quad (36)$$

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