Brief paper

Optimal linear state estimation over a packet-dropping network using linear temporal coding

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ABSTRACT

We consider the problem of linear minimum mean square error estimation for a discrete-time system over a packet-dropping network. In order to improve the estimation performance, different from the standard approach of sending the current measurement data, we choose sending a linear combination of the current measurement and the measurement collected at the previous time, a method called linear temporal coding. We assume the packet arrival sequence is unknown and the noise contained in the packet may come from sensor or communication channel. In an effort to cope with colored noise caused by measurement combination, after comparing with the classic state augmentation approach and measurement differencing approach, we derive a recursive estimation algorithm by means of orthogonal projection principle and innovation sequence approach. Our algorithm consists of two parts: smooth and estimate. For large measurement noise case, numerical example shows the benefit of using linear temporal coding strategy, compared with directly sending the current data. On the contrary, when communication noise plays the dominating role, for scalar system we prove there is no benefit to choose this scheme.

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1. Introduction

Thanks to the rapid development of wireless sensor and communication technology, recent years have witnessed a wealth of new applications of wireless sensor networks (WSNs) in many areas, including environmental monitoring and control, remote diagnostics and troubleshooting, factory automation and transportation (Mahalik, 2007). These systems are typically composed of spatially distributed low power sensors which are used to perform sensing, computing and communication. The information is exchanged among these spatially distributed devices through a shared communication network. This new area has attracted much research interest by different communities (Culler, Estrin, & Srivastava, 2004).

Compared with classic wired networks, WSNs provide many significant advantages, such as high flexibility, reduced wiring, low installation and maintenance costs, etc. At the same time, however, inserting a communication network introduces new research challenges, one of which is packet loss. The communication between sensors may be lost stochastically during the transmission, which is typically caused by transmission errors in physical network links, packets collisions or buffer overflows due to packets congestion (Hespanha, Naghshtabrizi, & Xu, 2007). Packet loss may severely degrade closed-loop system performance or even cause system instability. Thus, the effect of packet loss should not be neglected when designing the state estimator and controller. To compensate for packet loss, researchers have proposed many strategies.

Mesquita, Hespanha, and Nair (2009) suggested data retransmission which means a data packet is retransmitted until it is successfully received. This method is widely used in data network while the resulting delay may degrade performance of control system. At the same time, due to many practical reasons, network bandwidth is usually limited. Hence data retransmission is not applicable in networked control systems as in data network. Instead of retransmission, Mostofi and Murray (2009) designed another scheme to improve packet arrival rate, where a packet is used if the
corresponding signal-to-noise ratio of the channel is greater than a threshold and is discarded otherwise. In addition to these methods, one may also be interested in improving system performance while not taking extra communication resource. To facilitate the discussion, we first introduce the dynamical system considered in this paper

\begin{align}
    x_{k+1} &= Ax_k + \omega_k, \quad (1) \\
    y_k &= Cx_k + \nu_k, \quad (2)
\end{align}

where measurement \( y_k \) from the sensor is sent over a packet-dropping network to a remote estimator. Let \( y_k = 1 \) indicate that the packet successfully arrives at the estimator and \( y_k = 0 \) otherwise. Denote \( z_k^F \) as the information received by the estimator.

Sun, Xie, Xiao, and Soh (2008) considered a compensation mechanism which was first proposed by Sahebsara, Chen, and Shah (2007) and can be described as

\[ z_k^F = y_k x_k + (1 - y_k) z_{k-1}^F. \]

Using state augmentation approach and innovation sequence approach they derived the optimal linear algorithm for prediction, estimation and smooth. Liang, Chen, and Pan (2010) extended the idea of Sun et al. (2008) to deal with the case when unreliable transmissions exist in both sensor and control channels. Different from Liang et al. (2010) and Sun et al. (2008), Zhang, Yu, and Feng (2011) considered the optimal linear estimation for system with spatially distributed sensors and each sensor \( i \in \{1, \ldots, m\} \) sends the following data

\[ z_k^{F,i} = y_k x_k + (1 - y_k) z_{k-1}^{F,i}, \]

where \( r \in [0, 1] \) is an adjustable weighting parameter. Robinson and Kumar (2007) proposed a scheme, called linear temporal coding in which

\[ z_k^F = \gamma_k (\gamma_k + \beta y_{k-1}), \]

where \( \alpha, \beta \) are two weighting parameters. Assume \( \{y_k\} \) is available, they derived the optimal ratio between \( \alpha \) and \( \beta \) to minimize the estimation error covariance at time \( k + t \) when \( y_{k+1}, \ldots, y_{k+t} = (1, 0, \ldots, 0, 1) \) and demonstrated its advantages when compared with \( z_k^F = \gamma_k y_k \).

In Robinson and Kumar (2007), the estimator proposed has a mechanism to detect whether \( y_k = 1 \) or \( 0 \). In some applications, however, such \( y_k \) may not be available. For example, when there exists communication noise in the channel (Nahi, 1969), it is difficult for the estimator to infer whether packet dropping occurs. In this paper, we consider the following problem: when \( \{y_k\} \) is not available to the estimator, can we still improve the estimation performance using linear temporal coding strategy? If so, how does the packet arrival rate affect the optimal weighting parameter? Intuitively the more unreliable the network is, the more advantageous is using linear temporal coding since the estimator can partially recover the lost information \( y_{k-1} \) at time \( k \). In order to answer these questions, a first step is to derive a linear optimal estimation algorithm. To derive such a state estimation algorithm, most of the works (Liang et al., 2010; Robinson & Kumar, 2007; Sun et al., 2008; Zhang et al., 2011) chose state augmentation approach, which is a standard method in linear filtering. On the basis of the state augmentation form (6) and taking \( y_k \) as a system mode, Fletcher, Rangan, and Goyal (2004) gave another minimum mean square error estimator using Markov Jump Linear System (MJLS) theory when \( \{y_k\} \) is available. In the case when \( \{y_k\} \) is unknown, Costa (1994) derived the linear minimum mean square error (LMMSE) estimator for MJLS. More general cases were considered in their book (Costa, Fragoso, & Marques, 2005). The resulting estimation expressions via state augmentation approach, however, are often complicated, from which it is not easy to study the effect of weighting parameters on estimation performance.

Consider the discrete linear time-invariant system (1)–(2) (Fig. 1), where \( x_k \in \mathbb{R}^n \) is the state vector, \( y_k \in \mathbb{R}^m \) is the sensor measurement, \( \omega_k \in \mathbb{R}^n \) and \( \nu_k \in \mathbb{R}^m \) are zero-mean random vectors with \( E[\omega_k|\omega_{k-1}^t] = \delta_{ij} Q > 0 \), \( E[\nu_k|\nu_{k-1}^t] = \delta_{ij} R_n > 0 \) and \( E[\omega_k|\nu_{k-1}^t] = 0 \). Here \( \delta \) is the Kronecker delta function, i.e., \( \delta_{ij} = 0 \) if \( i \neq j \) and \( \delta_{ij} = 1 \) otherwise. The initial state is also assumed to be zero-mean with covariance \( P_0 \). We assume the pair \((A, C)\) is detectable and \((A, \sqrt{Q})\) is stabilizable.

Assume measurements collected by the sensor are sent over a packet-dropping network to the estimator. Recall that \( y_k \) indicates whether the packet sent at time \( k \) successfully arrives at the estimator or not, i.e., \( y_k = 1 \) means the packet arrives and \( y_k = 0 \) otherwise. Furthermore the packet arrival process \( \{y_k\} \) is assumed to be independent, identically distributed with \( E[y_k] = \lambda \). In this paper, we assume there is no acknowledgment to indicate whether the packet arrives or not.

Unlike the standard method of transmitting \( y_k \), we consider linear temporal coding and send \( z_k \) to the remote estimator, where \( z_k \) is defined as

\[ z_k = \alpha y_k + (1 - \alpha) z_{k-1}. \quad (3) \]

for a properly chosen \( \alpha \in [0, 1] \). In this paper, we also consider communication noise \( n_k \) (Dey, Leong, & Evans, 2009; Mostofi & Murray, 2009), which is zero-mean white with covariance \( R_n \) and independent of \( \{\omega_k\} \) and \( \{\nu_k\} \). Hence the information received by the estimator is \( z_k^F = y_k z_k + n_k \). We consider two cases in this paper:

- **T1**: Large measurement noise covariance \( R_n \gg R_d \) where the effect of \( n_k \) can be ignored. Thus

\[ z_k^F = y_k [\alpha C x_k + (1 - \alpha) C x_{k-1}] + [\alpha \nu_k + (1 - \alpha) \nu_{k-1}]. \quad (4) \]
of the augmented system. Expressions via the state augmentation approach, however, often fail to derive a LMMSE estimator under these two noise assumptions with unknown \( \gamma_k \). On that basis, we will also evaluate the influence of \( \alpha \) on the estimation performance.

3. Classic filtering algorithms for system with colored noise

For the large measurement noise case, one notes that \( z_k \) depends on both \( y_k \) and \( y_{k-1} \). Hence the noise term in \( z_k \) is no longer white. In this section, we state two well-known methods, i.e., state augmentation approach and measurement differencing approach, to tackle the correlated noise.

3.1. State augmentation approach (Anderson and Moore, 1979)

For system (1), one can augment the state as follows

\[
X_{k+1} = AX_k + W_k,
\]

where \( X_0 = [x_0; \ldots; x_{k-1}] \), \( W_k = [w_k; \ldots; w_{k-1}] \), \( A = [A; I; 0] \), \( C = [C; 0; 0] \), \( F = [\alpha(1-\alpha)] \). By introducing a dummy variable \( \xi_k \), we can further rewrite it as

\[
X_{k+1} = \bar{A}X_k + \bar{W}_k,
\]

where \( \bar{G} = [G; H \bar{F}] \), \( \bar{F} = [\gamma_k F(CX_k + \xi_k)] \).

Filtering for system (6) with no packet loss was discussed in Anderson and Moore (1979). In principle, there is no difficulty in dealing with system (6) with colored noise and packet loss. One can use classical Kalman filtering theory to design the LMMSE estimator for \( \{X_k; \xi_k\} \) as in Anderson and Moore (1979), and then combine that with the idea in Nahi (1969) to deal with unknown packet-dropping sequences. Taking \( \gamma_k \) as a mode, one can also use MJLS theory (Costa, 1994; Costa et al., 2005) to derive a LMMSE estimator for \( \{X_k; \xi_k\} \), which has been discussed by Fletcher et al. (2004) when \( \gamma_k \) is available. The resulting filter expressions via the state augmentation approach, however, often quite complex and not easy to analyze due to the high dimension of the augmented system.

3.2. Measurement differencing approach (Bryson & Henrikson, 1968; Henrikson, 1968)

Bryson and Henrikson (1968) proposed another scheme, the measurement differencing approach.

(1) Single step correlated measurement noise: Suppose the colored noise and the measurement equation of the system (1) can be written as

\[
\begin{align*}
\varepsilon_{k+1} & = \Psi \varepsilon_k + v_k, \\
y_k & = C \varepsilon_k + \varepsilon_k,
\end{align*}
\]

where \( v_k \) is white noise. Define the measurement difference term as

\[
\hat{y}_k \triangleq y_k - \Psi y_{k-1} = \bar{C} \varepsilon_{k-1} + (C \varepsilon_{k-1} + v_{k-1}),
\]

where \( \bar{C} \triangleq CA - \Psi C \). Now the new measurement noise \( C \varepsilon_{k-1} + v_{k-1} \) only correlates with the process noise, which can be easily dealt with. Henrikson extended the idea to more general systems in his Ph.D. Thesis (Henrikson, 1968).

(2) Two steps correlated measurement noise (Henrikson, 1968): Assume the measurement noise can be described as

\[
\begin{bmatrix}
\varepsilon_{k+1} \\
\varepsilon_k
\end{bmatrix} =
\begin{bmatrix}
\Psi_1 & I \\
0 & \Psi_2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_k \\
\varepsilon_k
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} v_k.
\]

Define

\[
\hat{y}_k \triangleq y_k - (\Psi_1 + \Psi_2) y_{k-1} + \Psi_2 \Psi_1 v_{k-2} = M_k + v_{k-2},
\]

where

\[
M_k \triangleq C \varepsilon_k + (\Psi_1 + \Psi_2) C \varepsilon_{k-1} + \Psi_2 \Psi_1 v_{k-2},
\]

which does not depend on the measurement noise. Hence the colored noise term is eliminated.

In our problem setup, the measurement equation can be written as

\[
z_k = FCX_k + [\alpha v_k + (1-\alpha) v_{k-1}].
\]

Here we consider a perfect network with all \( \gamma_k \)'s being 1 for illustrating the main idea. Given any \( \psi \), calculate the measurement differencing noise term as follows:

\[
\tilde{z}_k \triangleq z_k - \psi z_{k-1} = (F \bar{C} X_k - \psi F \bar{C} X_{k-1}) - (1-\alpha) \psi v_{k-2} + (1-\alpha) - \alpha \psi v_k - \alpha v_k.
\]

The colored noise term, however, cannot be removed using the measurement differencing approach.

4. Optimal linear estimator using innovation sequence approach

Unlike the two classic methods, in this section we derive a LMMSE estimator for system (1)–(3) using orthogonal projection principle and innovation sequence approach. Two important lemmas are introduced first, based on which lie our LMMSE estimation algorithm. In this paper, we define \( \text{Proj} \{X|Y\} \) as the LMMSE estimation of \( X \) given \( Y \), i.e., the projection of \( X \) on the space spanned by \( Y \). \( \mathbb{E} \{X\} \) stands for mathematical expectation of \( X \).

Lemma 4.1 (Anderson & Moore, 1979). Suppose the random variable \( \{X; Y\} \) has mean \( \{m_x; m_y\} \) and covariance \( \{\Sigma_{xx}, \Sigma_{xy}; \Sigma_{yx}, \Sigma_{yy}\} \), respectively. Then

\[
\text{Proj} \{X|Y\} = m_x + \Sigma_{xy} \Sigma_{yy}^{-1} (Y - m_y),
\]

with replacement of the inverse by a pseudo-inverse if necessary.

Lemma 4.2 (Anderson & Moore, 1979). Suppose that \( X, Y_1, \ldots, Y_k \) are jointly distributed, with \( Y_1, \ldots, Y_k \) mutually uncorrelated, i.e., \( \Sigma_{yjy} = 0 \)(\( i \neq j \)). Then

\[
\text{Proj} \{X|Y_1, \ldots, Y_k\} = \sum_{i=1}^{k} \text{Proj} \{X|Y_i\} - (k-1) m_x.
\]

We also define some symbols as follows: \( A \sim B \) implies that the variable \( A \) depends on \( B \); \( Z_k \) is the space spanned by the vectors \( z_1, \ldots, z_k \); \( \bar{Z}_k \) is the LMMSE estimation of \( Z_k \) given \( Z_{k-1} \); \( Z_k \) is the space spanned by \( \bar{Z}_1, \ldots, \bar{Z}_k \); \( \tilde{Z}_k \) is the innovation contained in \( \tilde{Z}_1, \ldots, \tilde{Z}_k \); \( \tilde{X}_k \) is the LMMSE estimation of \( \{X_k; \tilde{X}_k\} \); \( \tilde{X}_k \) is the LMMSE estimation of \( \{X_k; \tilde{X}_k\} \); \( \tilde{Z}_k \) is the covariance of \( X_{k-1} \) and \( \tilde{X}_k \). We also define their error covariances as

\[
\begin{align*}
P_x & \triangleq \mathbb{E} \{e_x c_x'\}, \\
P_k & \triangleq \mathbb{E} \{e_k c_k'\} \quad \text{for } 1 \leq k \leq n-1, \\
P_{x_k} & \triangleq \mathbb{E} \{e_{x_k} c_{x_k}'\}, \\
P_k & \triangleq \mathbb{E} \{e_y c_y'\} \quad \text{for } 1 \leq k \leq n-1. 
\end{align*}
\]
given $Z_k^{-1}$.

Similarly define $L_k \triangleq \text{Cov}(\hat{z}_k, \tilde{z}_k)$ and $M_k \triangleq \text{Cov}(\omega_k-1, \tilde{z}_k)$. Finally denote $S_k \triangleq \mathbb{E}[\tilde{x}_k' \tilde{x}_k]$ and $xx'$ is abbreviated as $(x)(x)'$ when the expression of $x$ is lengthy to state.

The central idea of our estimation algorithm is shown graphically in Fig. 2. Instead of computing $\hat{x}_k$ directly, one first gets the smooth of $x_{k-1}$, i.e., $\hat{x}_{k-1} \sim (\hat{x}_{k-1}, \tilde{z}_k)$. To derive $\hat{x}_k$, it is then better to use $\hat{x}_{k-1}^+ = \text{Proj} \{x_{k-1} | Z_{k-1} \}$ than $\hat{x}_{k-1} = \text{Proj} \{x_k | Z_{k-1}^{-1} \} = A \text{Proj} \{x_{k-1} | Z_{k-1}^{-1} \}$, since the former one has access to more information. Building on the above idea, we have the following result.

**Theorem 4.3.** A LMMSE estimation algorithm for system (1)–(3) with unknown $\gamma_k$ series consists of two steps:

1. **Smooth:**
   \[ \hat{x}_{k-1} = \text{Proj} \{x_{k-1} | Z_{k-1} \} \]
   \[ P_{k-1} = P_{k-1} - J_{k-1}^{-1} J_{k-1}' \] \[ (10) \]
2. **Estimate:**
   \[ \hat{\omega}_k = A \hat{x}_{k-1} + M_k L_k^{-1} \tilde{z}_k \]
   \[ P_k = AP_{k-1}A' + Q - (A \hat{\omega}_{k-1} + M_k) L_k^{-1} (A \hat{\omega}_{k-1} + M_k)' \] \[ (11) \]

**Proof.**
We first list some equations obtained by orthogonal projection principle (Anderson & Moore, 1979), which will be frequently used.

\[ \mathbb{E}[\omega_k \hat{z}_k] = 0 \]
\[ \mathbb{E}[\tilde{z}_k \hat{z}_k] = 0 \]
\[ \text{Proj} \{\tilde{z}_k | Z_{k-1}^{-1} \} = 0 \]

The last equality above holds true for the innovation sequence is derived via the so-called Gram–Schmidt procedure. Then calculate $L_k$ as

\[ L_k = \text{Cov}(\hat{z}_k, \tilde{z}_k) \]
\[ = \mathbb{E}[(\tilde{z}_k - \text{Proj} \{\tilde{z}_k | Z_{k-1}^{-1} \}) (\tilde{z}_k - \text{Proj} \{\tilde{z}_k | Z_{k-1}^{-1} \})] \]
\[ = \mathbb{E}[\tilde{z}_k \tilde{z}_k'] \]
\[ (12) \]

where the third equality is due to $\text{Proj} \{\tilde{z}_k | Z_{k-1}^{-1} \} = 0$. Similarly,

\[ J_{k-1} = \text{Cov}(\hat{x}_{k-1}, \tilde{z}_k) = \mathbb{E}[\hat{x}_{k-1} \tilde{z}_k] \]
\[ M_k = \text{Cov}(\omega_k-1, \tilde{z}_k) = \mathbb{E}[\omega_k-1 \tilde{z}_k] \]
\[ (13) \]

From Lemma 4.2, we have

\[ \hat{x}_{k-1}^+ = \text{Proj} \{x_{k-1} | Z_{k-1} \} \]
\[ = \text{Proj} \{x_{k-1} | Z_{k-1}^{-1} \} + \text{Proj} \{x_k | \tilde{z}_k \} - E[x_{k-1}] \]
\[ \hat{x}_{k-1}^+ = \text{Proj} \{x_{k-1} | Z_{k-1} \} \]
\[ = \text{Proj} \{x_{k-1} | Z_{k-1}^{-1} \} + \text{Proj} \{x_k | \tilde{z}_k \} - E[x_{k-1}] \]

where $\text{Proj} \{x_{k-1} | \tilde{z}_k \}$ can be computed via Lemma 4.1 as follows

\[ \text{Proj} \{x_{k-1} | \tilde{z}_k \} = E[x_{k-1}] + \text{Cov}(x_{k-1}, \tilde{z}_k) (\text{Cov}(\tilde{z}_k, \tilde{z}_k))^{-1} \tilde{z}_k \]

Recall the definition of $J_{k-1}$, $L_k$ and $\hat{x}_{k-1}$, we have

\[ \hat{x}_{k-1} = \hat{x}_{k-1} + L_k^{-1} \tilde{z}_k \]

Hence

\[ e_k = x_k - \tilde{\hat{x}}_k = e_k - J_k^{-1} L_k^{-1} \tilde{z}_k \]
\[ P_k = E[e_k e_k'] \]
\[ (15) \]

Combining (12) and (13) leads to the last equality above. Using again Lemmas 4.1 and 4.2 with the fact $\text{Proj} \{\omega_k-1 | Z_{k-1}^{-1} \} = 0$ yields

\[ \hat{\omega}_k = \text{Proj} \{x_k | Z_{k-1} \} \]
\[ = \text{Proj} \{x_k | Z_{k-1}^{-1} \} \]
\[ = \text{Proj} \{x_k | Z_{k-1}^{-1} \} \]
\[ = \text{Proj} \{x_k | Z_{k-1}^{-1} \} \]
\[ (16) \]

The last equality of (16) follows on using expression (10). From (12)-(14), together with (17) and $E[e_k \omega_k] = 0$, we get

\[ P_k = E[e_k e_k'] \]
\[ = A P_{k-1} A' + Q - (A \hat{\omega}_{k-1} + M_k) L_k^{-1} (A \hat{\omega}_{k-1} + M_k)' \]
\[ \Box \]

**Remark 4.4.** The estimation algorithm of Theorem 4.3 is relatively general, namely, it does not rely on whether the noise is white or colored. Calculated of $J_{k-1}$, $M_k$ and $L_k$ depends on the different models of $Z_k$, which will be presented in the next few subsections.

**Remark 4.5.** When there only exists correlated noise, innovation sequence approach has been considered by Jiang, Zhou, and Zhu (2010). Different from their work, our linear temporal coding strategy is novel in that we consider both smooth and estimate, and we consider estimation over an unreliable network.

### 4.1. Large measurement noise covariance: $R_e \gg R_n$

We rewrite the measurement Eq. (4) as

\[ z_k = \gamma_k (\tilde{\tilde{C}} \hat{x}_{k-1} + \hat{v}_k) \]
\[ (18) \]

where

\[ \tilde{\tilde{C}} \triangleq C [\alpha A + (1 - \alpha) I] \]
\[ \hat{v}_k \triangleq \alpha \omega_k - \alpha v_k + (1 - \alpha) v_{k-1} \]
\[ (19) \]

Note that $[\omega_k]$ and $[v_k]$ are still zero-mean noise, but the newly defined $[\hat{v}_k]$ is a colored noise sequence, i.e., $E[\hat{v}_k \hat{v}_k'] \neq 0$. We can write them in a compact form as

\[ E \left[ \begin{bmatrix} \omega_k & v_k \end{bmatrix} \right] = \begin{bmatrix} \mathbb{Q} & \mathbb{O} \\ \mathbb{O}' & \mathbb{R} \end{bmatrix} \]
\[ (21) \]

where

\[ \mathbb{O} \triangleq \alpha Q C, \quad \mathbb{R} \triangleq (1 - \alpha) R_e \]
\[ \tilde{\mathbb{R}} \triangleq \alpha^2 Q C + [\alpha^2 + (1 - \alpha)^2] R_e \]

Now we derive explicit expressions for $J_{k-1}$, $M_k$ and $L_k$. 

---

2 The covariance here cannot be deemed as the conditional covariance since we use LMMSE estimation $\text{Proj} \{x_{k-1} | Z_{k-1}^{-1} \}$ instead of conditional expectation $E[x_{k-1} | Z_{k-1}^{-1}]$. 

For the unknown Bernoulli random variable $\gamma_k$, we have

$$E[(\gamma_k - \lambda)^2] = \lambda(1 - \lambda),$$
$$E[\gamma_k] = E[E[\gamma_k^i]] = \lambda, \quad E[\gamma_i|\gamma] = \lambda^2 \quad (i \neq j).$$

Compute $\hat{z}_k^*$ as

$$\hat{z}_k^* = \text{Proj.}[\hat{e}_k^*L_{k-1}^{-1}\hat{z}_k^* - \hat{v}_k] = \lambda[\hat{c}_k^* - \text{Proj.}[\hat{v}_k]\hat{z}_k^*].$$

Since

$$\hat{z}_k^* = (\gamma_0; \alpha_0, \ldots, \alpha_{k-2}; v_0, \ldots, v_{k-1}; \gamma_1, \ldots, \gamma_{k-1}),$$
we have Proj.$[\alpha_{k-1}\hat{z}_k^*] = 0$, Proj.$[\hat{v}_k\hat{z}_k^*] = 0$ and $E[\hat{v}_k - (\hat{z}_k^*)'] = 0$. By Lemma 4.1,

$$\text{Proj.}[\hat{v}_k\hat{z}_k^*] = (1 - \alpha)[\text{Proj.}[\hat{v}_k\hat{z}_k^*] + \text{Cov}(v_{k-1}, \hat{z}_k^*)]$$
$$= (1 - \alpha)E[\hat{v}_k\hat{z}_k^*]L_{k-1}^{-1}\hat{z}_k^*.$$

Since

$$E[\hat{v}_k\hat{z}_k^*] = E[\gamma_1\hat{c}_k^* - \hat{v}_k\hat{z}_k^*] = E[\gamma_1\hat{c}_k^* - \alpha\hat{v}_k\hat{z}_k^*] = \lambda\alpha\hat{r}_k,$$

Substituting (24)–(25) into (22) yields

$$\hat{z}_k^* = \lambda[\hat{c}_k^* - \lambda\hat{r}_kL_{k-1}^{-1}\hat{z}_k^*].$$

As shown in (10), in order to derive $\hat{z}_k^*$, we calculate $J_k$ in (13) as

$$J_k = E[\hat{e}_k^-\hat{z}_k^-] = \lambda[\hat{c}_k^- + E[\hat{e}_k^-\hat{v}_k^-]],$$

where we use the orthogonality condition $E[\hat{e}_k^-\hat{z}_k^-] = 0$ and $E[\hat{e}_k^-\hat{v}_k^-] = 0$. Since

$$E[\hat{e}_k^-\hat{v}_k^-] = -(1 - \alpha)E[\hat{z}_k^-\hat{v}_k^-]$$

we derive the expression for $\hat{z}_k^*$ (or $\hat{x}_k^*$) first. As Eq. (16) shows, we need to calculate $M_k$ in (14) as

$$M_k = E[\hat{e}_k^-\hat{z}_k^-] = E[\hat{e}_k^-\hat{v}_k^-].$$

Then

$$E[\hat{e}_k^-\hat{v}_k^-] = -(1 - \alpha)E[\hat{z}_k^-\hat{v}_k^-]$$

by (25). Combining (28) with (30) yields

$$J_k = \lambda P_{k-1}^-\hat{c}_k^- - \lambda^2(A_{k-2}^- + M_{k-1}^-)L_{k-1}^{-1}\hat{r}_k^-.$$

Now let us turn back to calculate $L_k$. Employing again the orthogonal condition $E[\hat{e}_k^-\hat{z}_k^-] = 0$, we have

$$S_k = E[\hat{e}_k^*\hat{z}_k^*] = AS_k^-A' + Q,$$

$$P_{k-1} = E[\hat{e}_k^-\alpha_{k-1}'] = E[\hat{e}_k^-\hat{v}_k^-].$$

As

$$\hat{z}_k = \gamma_k\hat{c}_k + (\gamma_k - \lambda)\hat{c}_k^* + \gamma_k\hat{v}_k - \lambda^2\hat{r}_kL_{k-1}^{-1}\hat{z}_k^*,$$

one can easily verify that

$$E[(\gamma_k\hat{c}_{k-1})'] = \lambda\hat{c}_{k-1}\hat{c}_k',$$
$$E[(\gamma_k - \lambda)\hat{c}_k|\gamma_{k-1}] = \lambda(1 - \lambda)\hat{c}(S_{k-1} - P_k)\hat{c}_k',$$
$$E[(\gamma_k\hat{v}_k)'] = \lambda\hat{r}_k,$$
$$E[(-\lambda^2\hat{r}_kL_{k-1}^{-1}\hat{z}_k)'] = \lambda^2\hat{r}_kL_{k-1}^{-1}\hat{r}_k.$$

From the orthogonal principle, we have

$$E[(\gamma_k\hat{c}_{k-1})|\gamma_{k-1}] = 0,$$
$$E[(\gamma_k\hat{c}_{k-1})|\gamma_{k-1}] = 0.$$

Since neither $\hat{z}_k$ nor $\hat{z}_k^*$ relies on $\gamma_k$, and $E[\gamma_k - \lambda] = 0$, we obtain

$$E[(\gamma_k\hat{v}_k)(\gamma_k - \lambda)\hat{c}_k^*|\gamma_{k-1}] = E[(-\lambda^2\hat{r}_kL_{k-1}^{-1}\hat{z}_k^*)|\gamma_{k-1}] = 0.$$

In view of (20), (23) and (25), we have

$$E[\hat{v}_k\hat{z}_k^-] = (1 - \alpha)E[\hat{v}_k\hat{z}_k^-] = \lambda\hat{r}_k.$$

Hence

$$E[(\gamma_k\hat{v}_k)(\gamma_k - \lambda)\hat{c}_k^*|\gamma_{k-1}] = -2\lambda^2\hat{r}_kL_{k-1}^{-1}\hat{r}_k.$$

From (30), one has

$$E[\hat{e}_k^-\hat{v}_k^-] = -E[\hat{e}_k^-\hat{v}_k^-] = \lambda(A_{k-1}^- + M_{k-1}^-)L_{k-1}^{-1}\hat{r}_k,$$

where $\lambda$ comes from $\gamma_{k-1}$ and does not relate to $\gamma_k$. Therefore

$$E[(\gamma_k\hat{c}_{k-1})(\gamma_k\hat{v}_k)'] + E[(\gamma_k - \lambda)\hat{c}_k^*|\gamma_{k-1}]$$

$$E[(\gamma_k\hat{v}_k)|\gamma_{k-1}] + E[(\gamma_k\hat{v}_k)|\gamma_{k-1}] = \lambda^2\hat{c}(A_{k-2}^- + M_{k-1}^-)L_{k-1}^{-1}\hat{r}_k.$$

Combining all the results above yields

$$L_k = E[\hat{z}_k\hat{z}_k^*]$$

$$= \lambda[\hat{c}_{k-1}\hat{c}_k^* + (1 - \lambda)\hat{c}(S_{k-1} - P_k)\hat{c}_k^* + \hat{r}_k - \lambda^2\hat{c}(A_{k-2}^- + M_{k-1}^-)L_{k-1}^{-1}\hat{r}_k$$

$$- \lambda^2\hat{r}_kL_{k-1}^{-1}(A_{k-2}^- + M_{k-1}^-)\hat{c}_k^*].$$

Up to this point, we have obtained all the parameters for the filter expression (10)–(11) when $R_\gamma \gg R_n$. For the initial conditions, one can set $\gamma_0 = 0$ and $P_0 = P_{\lambda}$. Suppose we do not send packet at time $k = 1$, which means $\gamma_1 = 0, \hat{x}_1 = A_0, P_1 = A_0A' + Q$, then $S_2 = \gamma_2\hat{z}_2 - \lambda\hat{c}\hat{x}_1, J_1 = \lambda\hat{c}\hat{c}'$ and $L_2 = \lambda[\hat{c}\hat{c}' + (1 - \lambda)\hat{c}(S_1 - P_1)\hat{c}' + \hat{r}_1].$ The complete estimation algorithm is given by Algorithm 1 and denoted as

$$\hat{\lambda}(\hat{x}_k, P_k) = \Phi(\hat{x}_k, P_k, S_k, \gamma_k, \lambda, \hat{r}_k, \hat{r}_1).$$

4.2. Large communication noise covariance: $R_\nu \ll R_n$

Rewrite Eq. (5) as

$$\hat{z}_k^* = \gamma_k\hat{c}_k + \hat{\pi}_k,$$

where

$$\hat{\pi}_k = \lambda\hat{c}\hat{v}_k + n_k,$$

and $\hat{c}$ is same as in (19). Clearly $\{\gamma_{k-1}\}$ and $\{\hat{\pi}_k\}$ are zero-mean white with

$$E\left[\begin{bmatrix} \hat{\pi}_k - \lambda\hat{c}\hat{v}_k \\ \hat{\pi}_k \end{bmatrix} \right] = \left[ \begin{bmatrix} Q & \lambda\hat{c}\hat{Q}' \\ \lambda^2\hat{c}\hat{Q}' & R_n \end{bmatrix} \right].$$

(37)
Algorithm 1 Estimation algorithm for \( R_0 \gg R_n \)

1: Initial conditions: \( \dot{R}_0 \triangleq \alpha_2^2 \text{CQ}^2 + \alpha_2^2 + (1 - \alpha)^2 R_0 \), \( R_1 \triangleq \alpha(1 - \alpha) R_0 \), \( \Omega \triangleq \alpha_2^2 \text{CQ}^2, \dot{\ddot{C}} \triangleq \alpha \text{C}(\alpha A + (1 - \alpha) I) \), \( x_0 \sim \mathcal{N}(0, P_0) \), \( S_0 = \Phi_0 \), \( \tilde{x}_0 = 0 \), \( P_0 = \Phi_0 \), \( \tilde{x}_1 = \tilde{A}\tilde{x}_0 \), \( P_1 = \Phi P_0 + Q + \tilde{U} \), \( J_1 = \lambda P_1 \tilde{C} \) and \( \tilde{z}_1 = \beta \tilde{C}(P_1 \tilde{C}^T + \tilde{R}_0) \), \( \tilde{S}_1 \), \( \tilde{P}_2 \), \( S_2 \), \( \tilde{S}_2 \) are calculated from step 5–8.

2: LMMSE estimation of \( \tilde{z}_k \) given \( Z_{k-1} \)
\[
\tilde{z}_k = \frac{1}{L_k} = \frac{1}{\lambda} \tilde{C}(P_{k-1} \tilde{C} + \tilde{R}_0) \}
\]

3: Covariance of \( x_k-1 \) and \( z_k \) given \( Z_{k-1} \)
\[
J_{k-1} = \lambda \tilde{P}_{k-1} \tilde{C} \times \lambda \tilde{A}(J_{k-2} + M_k \tilde{R}) L_{k-1} \tilde{R}_1 \times 10^3 \]

4: Variance of \( \tilde{z}_k \) and \( \tilde{z}_k \) given \( Z_{k-1} \)
\[
L_k = \lambda \tilde{C}(P_{k-1} \tilde{C} + \tilde{R}_0) \times 10^3 \]

5: Smooth step
\[
\begin{align*}
\hat{x}_{k-1}^1 &= \hat{x}_{k-1}^1 + J_{k-1}^{-1} \{ z_k - \hat{z}_{k-1} \} \\
\tilde{P}_{k-1}^1 &= \tilde{P}_{k-1}^{-1} - L_{k-1}^{-1} \tilde{R}_1
\end{align*}
\]

6: Covariance of \( \omega_k-1 \) and \( \tilde{z}_k \) given \( Z_{k-1} \)
\[
M_k = \Xi \Omega \]

7: Estimate step
\[
\hat{x}_k = \tilde{A}\hat{x}_{k-1}^1 + M_k \tilde{R}_k \{ z_k - \hat{z}_{k-1} \}, \quad \tilde{P}_k = \tilde{P}_{k-1}^1 - \tilde{R}_1 \tilde{R}_k \{ z_k - \hat{z}_{k-1} \}
\]

8: Recursion for \( S_k \| E[x_k x_k^T] \)
\[
S_k = \tilde{A}S_{k-1}^{-1} + Q
\]

Remark 4.7. Both in Algorithms 1 and 2, one may find that \( [P_k] \) rely on \( [S_k] \) which comes from the unknown \( [\gamma_k] \). The same issue arises in Costa (1994), Nahi (1969) and Sun et al. (2008). Suppose the system is unstable, \( [S_k] \) diverges. In other words, \( L_{k-1} \) tends to zero and the estimation becomes open-loop prediction as \( k \rightarrow \infty \). To overcome this difficulty, one may consider adding a reliable control channel to limit the state, e.g.,
\[
x_{k-1} = \tilde{A}x_k - BG \tilde{x}_k + \omega_k.
\]

Similar idea was considered by Gupta, Hassibi, and Murray (2007) and Tatikonda and Mitter (2004).

5. Examples

Consider the following illustrative example when \( R_0 \gg R_n: A = [0.98 0.01; 0.97]; C = [1 0; 0 1]; Q = [0.04 0; 0.04]; \phi_0 = [2.4 0; 0 2.4]. \) To verify the filtering algorithms derived in this paper we choose \( \alpha = 0.8, \lambda = 0.75 \) and \( P_0 = \text{diag}(0.3, 0.3). \) The state and its estimate for every component is shown in Fig. 3 and the trace of estimation error covariance is shown in Fig. 4.

To study the influence of \( \alpha \) on estimation performance, let \( P_\infty = \lim_{\alpha \to \infty} P_\infty \) and \( S_\infty = \lim_{\alpha \to \infty} S_\infty \). Then we are interested in the performance improvement rate of linear temporal coding strategy compared with usual scheme \( \hat{z}_k = \gamma_k \tilde{y}_k \), defined as follows
\[
\beta = \frac{\text{Tr}(P_\infty(\alpha = 1)) - \text{Tr}(P_\infty(\alpha \to \infty))}{\text{Tr}(P_\infty(\alpha = 1))}.
\]

Fig. 5 shows the relationship between \( \text{Tr}(P_\infty) \) and \( \alpha \) for \( \lambda = 0.3 \), where \( \alpha^* = 0.543 \) and corresponding \( \beta^* \) is 9.1%. For general packet arrival rate, \( \alpha^* \) versus \( \alpha \) is shown in Fig. 6 and \( \beta^* \) versus \( \lambda \) is given by Fig. 7. Fig. 6 implies when network is more unreliable, one should increase the weight of \( y_k-1 \). On the other hand, one expects that compared with \( y_k-1 \), the current measurement \( y_k \) always contributes more for decreasing the estimation error. In other words, in \( z_k = \alpha y_k + (1 - \alpha) y_k-1 \), we have \( \alpha^* \geq 0.5 \).
Fig. 3. State and estimate for each component.

Fig. 4. Trace of the estimation error covariance.

Let $\frac{\partial P_\infty}{\partial \alpha} = 0$, with some direct calculation, we get

$$
\alpha^* = \frac{Q[\lambda P_\infty + (1 - \lambda)S_\infty] + \frac{R_n}{2\alpha} [A^2 P_\infty - AP_\infty + Q]}{Q[(1 - \lambda)A(P_\infty - S_\infty) + [\lambda P_\infty + (1 - \lambda)S_\infty]]}
= 1 + \frac{\frac{R_n}{2\alpha} [A^2 P_\infty - AP_\infty + Q] + (1 - \lambda)AQ(S_\infty - P_\infty)}{Q([\lambda I + (1 - \lambda)A]P_\infty + (1 - \lambda)(I - A)S_\infty)].}
$$

(39)

From Algorithm 2, we have

$$
P_\infty = AP_\infty A' + Q - (A_\infty + M_\infty) L^{-1} (A_\infty + M_\infty)'.
$$

(40)

For a stable scalar system,

$$
A^2 P_\infty + Q = P_\infty + \frac{(A_\infty + M_\infty)^2}{L_\infty} \geq P_\infty > AP_\infty,
$$

which means $A^2 P_\infty - AP_\infty + Q > 0$. From $S_\infty = AS_\infty A' + Q$ and (40), we have $S_\infty - P_\infty > 0$. From (39) and recall that $\alpha \in [0, 1]$, we get $\alpha^* = 1$. In other words, different from the $R_n \gg R$ case, when $R_n \ll R$, there is no benefit to use linear temporal coding strategy. Consider the following system whose parameters are: $A = 0.97$, $C = 1$, $Q = 0.04$, $R_n = 2.4$, $\Pi_0 = 0.3$. When $\lambda = 0.3$, the relationship between $P_\infty$ and $\alpha$ is shown in Fig. 8.

6. Conclusion

In this paper, we considered the LMMSE estimation problem over an unreliable network. We used the linear temporal coding strategy to improve the estimation performance when the sensor data is communicated to a remote estimator over a packet-dropping network. To cope with colored noise from measurement combination, after comparing with the standard state-augmentation approach and the measurement differencing approach, we choose orthogonal projection principle and innovation sequence approach to get the recursive estimation algorithm.
For large measurement noise variance case we show the optimal weight parameter relies on packet arrival rate via an illustrative example. By contrast, for scalar system we prove there is no benefit to choose this strategy for the large communication noise case. There are many interesting directions for continuing this work. When $R_v \gg R_n$, derive a closed-form expression for $\alpha^*$ with respect to system parameters and network condition; find a general scheme that outperforms the one developed in this paper, e.g., $z_k = q_1y_k + q_{d-1}y_{k-1} + \cdots + q_{d-1}y_{k-d}$; when $R_v \ll R_n$, extend the proof to higher-order system and experimentally evaluate the theory developed in this paper.

References


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