

Complex Matrix Decomposition and Quadratic Programming

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This paper studies the possibilities of the linear matrix inequality characterization of the matrix cones formed by nonnegative complex Hermitian quadratic functions over specific domains in the complex space. In its real-case analog, such studies were conducted in Sturm and Zhang [Sturm, J. F., S. Zhang. 2003. On cones of nonnegative quadratic functions. *Math. Oper. Res.* **28** 246–267]. In this paper it is shown that stronger results can be obtained for the complex Hermitian case. In particular, we show that the matrix rank-one decomposition result of Sturm and Zhang [Sturm, J. F., S. Zhang. 2003. On cones of nonnegative quadratic functions. *Math. Oper. Res.* **28** 246–267] can be strengthened for the complex Hermitian matrices. As a consequence, it is possible to characterize several new matrix co-positivity cones (over specific domains) by means of linear matrix inequality. As examples of the potential application of the new rank-one decomposition result, we present an upper bound on the lowest rank among all the optimal solutions for a standard complex semidefinite programming (SDP) problem, and offer alternative proofs for a result of Hausdorff [Hausdorff, F. 1919. Der Wertvorrat einer Bilinearform. *Mathematische Zeitschrift* **3** 314–316] and a result of Brickman [Brickman, L. 1961. On the field of values of a matrix. *Proc. Amer. Math. Soc.* **12** 61–66] on the joint numerical range.

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1. Introduction. The aim of this paper is to extend the results on the cone of nonnegative quadratic functions, studied by Sturm and Zhang in [9], from the real-valued to the complex domains. Sturm and Zhang [9] developed a matrix rank-one decomposition technique, which is a key technique in their approach to establish the linear matrix inequality representability of a class of matrix cones of nonnegative quadratic functions. It turns out that in the case of complex (Hermitian) quadratic forms, the rank-one decomposition result can actually be improved. In particular, we show in this paper that it is possible to find a rank-one decomposition for a positive semidefinite Hermitian matrix such that the inner-product between any of the rank-one matrices and two prescribed Hermitian matrices are constant, respectively. As a comparison, in the real case, the inner-product of these rank-one matrices and only one prescribed matrix can be made constant in general.

The organization of this paper is as follows. Section 2 studies such matrix rank-one decompositions, and §3 is devoted to the description of the cone of nonnegative complex quadratic functions. The results of Sturm and Zhang [9] can be applied to solve quadratic optimization problems, as shown in Ye and Zhang [11]. Some of the results can be strengthened for the Hermitian forms, due to the results established in §§2 and 3. Section 4 is devoted to the complex quadratic programming problem. In §5, we study the rank of optimal solutions for a standard complex SDP, in light of the new rank-one decomposition result. Finally, in §6 we investigate some interesting relationships between the rank-one decomposition theorem and the joint numerical range.

Notation. Throughout, we denote by \bar{a} the conjugate of a complex number a , and by \mathbf{C}^n the space of n -dimensional complex vectors. For a given vector $z \in \mathbf{C}^n$, z^H denotes the conjugate transpose of z . The space of $n \times n$ real symmetric and complex Hermitian matrices are denoted by \mathcal{S}^n and \mathcal{H}^n , respectively. For a matrix $Z \in \mathcal{H}^n$, we write $\text{Re } Z$ and $\text{Im } Z$ for the real and imaginary parts of Z , respectively. Matrix Z being Hermitian implies that $\text{Re } Z$ is symmetric and $\text{Im } Z$ is skew-symmetric. We denote by \mathcal{S}_+^n (\mathcal{S}_{++}^n) and \mathcal{H}_+^n (\mathcal{H}_{++}^n) the cones of real symmetric positive semidefinite (positive definite) and complex Hermitian positive semidefinite (positive definite) matrices, respectively. The notation $Z \succeq (>0)$ means that Z is positive semidefinite (positive definite). For two complex matrices Y and Z , their inner product $Y \bullet Z = \text{Re}(\text{tr } Y^H Z) = \text{tr}[(\text{Re } Y)^T (\text{Re } Z) + (\text{Im } Y)^T (\text{Im } Z)]$, where tr denotes the trace of a matrix and T denotes the transpose of a matrix. For a square matrix M , $\text{diag}(M)$ stands for a column vector whose elements are diagonal components of M .

2. A rank-one decomposition of Hermitian positive semidefinite (PSD) matrices. Let $X \in \mathcal{S}^n$ be a real symmetric positive semidefinite matrix, and $A \in \mathcal{S}^n$ be a real symmetric matrix. It follows by Sturm and Zhang [9] that there is a rank-one decomposition of X :

$$X = \sum_{j=1}^r x_j x_j^T \quad \text{such that} \quad x_j^T A x_j = \frac{A \bullet X}{r}, \quad \text{for } j = 1, \dots, r,$$

where $r = \text{rank } X$.

Now we shall show that in the Hermitian case the decomposition result can be extended to two matrices.

THEOREM 2.1. *Suppose that $Z \in \mathcal{H}^n$ is a complex Hermitian positive semidefinite matrix of rank r , and $A, B \in \mathcal{H}^n$ be two given Hermitian matrices. Then, there is a rank-one decomposition of Z ,*

$$Z = \sum_{j=1}^r z_j z_j^H,$$

such that

$$z_j^H A z_j = \frac{A \cdot Z}{r}, \quad z_j^H B z_j = \frac{B \cdot Z}{r}, \quad j = 1, \dots, r.$$

PROOF. It follows from Sturm and Zhang [9, Corollary 4] that there is a decomposition of Z

$$Z = \sum_{j=1}^r u_j u_j^H \quad \text{such that} \quad u_j^H A u_j = \frac{A \cdot Z}{r}, \quad \text{for } j = 1, \dots, r.$$

If $u_j^H B u_j = B \cdot Z / r$ for any $j = 1, \dots, r$, then we are done. Otherwise, there should be two indices, say 1 and 2, such that

$$u_1^H B u_1 > \frac{B \cdot Z}{r} \quad \text{and} \quad u_2^H B u_2 < \frac{B \cdot Z}{r}.$$

Denote $u_1^H A u_1 = \gamma_1 e^{i\alpha_1}$ and $u_2^H A u_2 = \gamma_2 e^{i\alpha_2}$. Let $w = \gamma e^{i\alpha} \in \mathbf{C}$ with $\alpha = \alpha_1 + \pi/2$ and $\gamma > 0$ a root of the real quadratic equation in terms of x :

$$\left(u_1^H B u_1 - \frac{B \cdot Z}{r} \right) x^2 + 2(\gamma_2 \sin(\alpha_2 - \alpha_1))x + u_2^H B u_2 - \frac{B \cdot Z}{r} = 0. \quad (1)$$

Remark that since $u_1^H B u_1 - B \cdot Z / r > 0$ and $u_2^H B u_2 - B \cdot Z / r < 0$, the above equation must have two real roots with opposite signs, and γ is taken to be the positive root.

Set

$$v_1 = (w u_1 + u_2) / \sqrt{1 + \gamma^2}, \quad v_2 = (-u_1 + \bar{w} u_2) / \sqrt{1 + \gamma^2}.$$

It is easy to verify that

$$v_1 v_1^H + v_2 v_2^H = u_1 u_1^H + u_2 u_2^H. \quad (2)$$

Moreover,

$$\begin{aligned} (1 + \gamma^2) v_1^H A v_1 &= (\bar{w} u_1^H + u_2^H) A (w u_1 + u_2) \\ &= \gamma^2 u_1^H A u_1 + u_2^H A u_2 + \bar{w} u_1^H A u_2 + w u_2^H A u_1 \\ &= \gamma^2 u_1^H A u_1 + 2\gamma \gamma_1 \operatorname{Re} e^{i(\alpha_1 - \alpha)} + u_2^H A u_2 \\ &= \gamma^2 u_1^H A u_1 + u_2^H A u_2 \\ &= (\gamma^2 + 1) \frac{A \cdot Z}{r}, \end{aligned}$$

which amounts to $v_1^H A v_1 = A \cdot Z / r$. Therefore, $v_2^H A v_2 = A \cdot Z / r$. At the same time, we have

$$\begin{aligned} (1 + \gamma^2) v_1^H B v_1 &= (\bar{w} u_1^H + u_2^H) B (w u_1 + u_2) \\ &= \gamma^2 u_1^H B u_1 + u_2^H B u_2 + 2 \operatorname{Re}(\bar{w} u_1^H B u_2) \\ &= \gamma^2 u_1^H B u_1 + 2\gamma \gamma_2 \sin(\alpha_2 - \alpha_1) + u_2^H B u_2 \\ &= (1 + \gamma^2) \frac{B \cdot Z}{r}, \end{aligned}$$

where in the last equality we use the fact that γ solves (1).

Due to (2), by letting $z_1 = v_1$ we get

$$Z - z_1 z_1^H = Z - v_1 v_1^H = \sum_{j=2}^r u_j u_j^H \succeq 0.$$

We conclude that $z_1^H A z_1 = A \cdot Z/r$ and $z_1^H B z_1 = B \cdot Z/r$. Note that $\text{rank}(Z - z_1 z_1^H) = r - 1$ and $v_2^H A v_2 = u_j^H A u_j = A \cdot Z/r$ for $j = 3, \dots, r$. Repeating this process, we obtain a rank-one decomposition for Z :

$$Z = \sum_{j=1}^r z_j z_j^H,$$

with the property that $z_j^H A z_j = A \cdot Z/r$ and $z_j^H B z_j = B \cdot Z/r$, $j = 2, \dots, r$. \square

Remark that when we construct the new vectors v_1, v_2 by choosing $w \in \mathbf{C}$ in the above proof, there are in fact two independent quantities: the argument and the modulus of w , which makes it possible to involve two Hermitian matrices A and B . Exactly for this reason this result does not hold for the real matrices.

An immediate corollary follows.

COROLLARY 2.1. *Let $A, B \in \mathcal{H}^n$ be two arbitrary matrices. Let $Z \in \mathcal{H}^n$ be a positive semidefinite matrix of rank r . Suppose that $A \cdot Z \succeq_1 0$ and $B \cdot Z \succeq_2 0$, where $\succeq_1, \succeq_2 \in \{\geq, \leq, =, >, <\}$. Then there is a rank-one decomposition for Z*

$$Z = \sum_{j=1}^r z_j z_j^H$$

such that $z_j^H A z_j \succeq_1 0$ and $z_j^H B z_j \succeq_2 0$ for all $j = 1, \dots, r$.

3. Cone of complex nonnegative quadratic functions. In this section, by employing the rank-one decomposition theorem, we aim at characterizing by linear matrix inequality various matrix cones formed by complex Hermitian quadratic functions that are nonnegative over a specific domain.

Let $D \subset \mathbf{C}^n$ be a given set. Consider all Hermitian matrices that are copositive over D , i.e.,

$$\mathcal{C}_+(D) = \{Z \in \mathcal{H}^n: z^H Z z \geq 0, \forall z \in D\}. \quad (3)$$

Clearly, $\mathcal{C}_+(D)$ is a closed convex cone in \mathcal{H}^n . The cone of all complex quadratic functions that are nonnegative over D is defined by

$$\mathcal{F}\mathcal{C}_+(D) = \left\{ \begin{bmatrix} c & b^H \\ b & A \end{bmatrix} \in \mathcal{H}^{n+1}: c + 2\text{Re}(b^H z) + z^H A z \geq 0, \forall z \in D \right\}. \quad (4)$$

For a quadratic function $q(z) = z^H A z + 2\text{Re}(b^H z) + c$, we introduce its matrix representation as

$$q(z) = M(q(\cdot)) \cdot \begin{bmatrix} 1 & z^H \\ z & z z^H \end{bmatrix},$$

where

$$M(q(\cdot)) = \begin{bmatrix} c & b^H \\ b & A \end{bmatrix}.$$

Apparently, $q(z) \geq 0$ for all $z \in D$ if and only if $M(q(\cdot)) \in \mathcal{F}\mathcal{C}_+(D)$. The homogenization of a given set D is defined by

$$\mathcal{H}(D) = \text{cl} \left\{ \begin{bmatrix} s \\ z \end{bmatrix} \in \mathfrak{R}_{++} \times \mathbf{C}^n: z/s \in D \right\},$$

where cl stands for the closure operation.

For a given set D we denote by $\text{conv}(D)$ the convex hull of D , i.e., the intersection of all convex sets containing D , and by $\text{cone}(D)$ the convex cone hull of D , i.e., the intersection of all convex cones containing D . The dual cone of \mathcal{K} in \mathcal{H}^n is defined as $\mathcal{K}^* = \{Y \in \mathcal{H}^n: Y \cdot Z \geq 0, \forall Z \in \mathcal{K}\}$. Following similar arguments as in Sturm and Zhang [9], the next two propositions are immediate.

PROPOSITION 3.1. *It holds that*

$$\mathcal{C}_+(D) = (\text{cone}\{z z^H: z \in D\})^*$$

and

$$\text{cl cone}\{z z^H: z \in D\} = \text{cone}\{z z^H: z \in \text{cl } D\}.$$

PROPOSITION 3.2. For any nonempty set $D \subset \mathbb{C}^n$, there holds

$$\mathcal{F}\mathcal{C}_+(D) = \mathcal{C}_+(\mathcal{H}(D)) = \mathcal{C}_+\left(\bigcup_{u \in \mathbb{C}, |u|=1} (u\mathcal{H}(D))\right).$$

In what follows, we shall give a characterization of $\mathcal{C}_+(D)$ where D is defined by

$$D = \{z \in \mathbb{C}^n: z^H A z \geq 0, z^H B z \geq 0\}.$$

Our next result follows directly from Theorem 2.1.

THEOREM 3.1. Suppose that $A, B \in \mathcal{H}^n$ and $D = \{z \in \mathbb{C}^n: z^H A z \geq 0, z^H B z \geq 0\}$. Then, we have

$$\text{cone}\{zz^H: z \in D\} = \text{conv}\{zz^H: z \in D\} = \{Z \succeq 0: A \bullet Z \geq 0, B \bullet Z \geq 0\}.$$

PROOF. Obviously, $\text{conv}\{zz^H: z \in D\} \subseteq \text{cone}\{zz^H: z \in D\}$. The equality follows from the observation that $\text{conv}\{zz^H: z \in D\}$ is itself a convex cone. That $\text{conv}\{zz^H: z \in D\} \subseteq \{Z \succeq 0: A \bullet Z \geq 0, B \bullet Z \geq 0\}$ is clear, thanks to Carathéodory's theorem. Now, $\{Z \succeq 0: A \bullet Z \geq 0, B \bullet Z \geq 0\} \subseteq \text{conv}\{zz^H: z \in D\}$ follows from Theorem 2.1 by construction. \square

The dual form of Theorem 3.1 is also known as the S-Lemma, which we shall present below. The result was first shown by Fradkov and Yakubovich [4], though their proof was totally different.

THEOREM 3.2. Suppose that $A, B \in \mathcal{H}^n$ and $D = \{z \in \mathbb{C}^n: z^H A z \geq 0, z^H B z \geq 0\}$. Furthermore, suppose that there is $z_0 \in \mathbb{C}^n$ such that $z_0^H A z_0 > 0, z_0^H B z_0 > 0$, then

$$\{Z \in \mathcal{H}^n: z^H Z z \geq 0, \forall z \in D\} = \{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

PROOF. It follows Proposition 3.1, the bipolar theorem, and Theorem 3.1 that

$$\begin{aligned} \mathcal{C}_+(D) &= \{Z: z^H Z z \geq 0, \forall z \in D\} \\ &= (\text{conv}\{zz^H: z \in D\})^* \\ &= (\{Z \succeq 0: A \bullet Z \geq 0, B \bullet Z \geq 0\})^* \\ &= \text{cl}\{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\}. \end{aligned}$$

It remains to show that $\{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\}$ is a closed set. To this end, take any sequence $Z_k, s_k \geq 0, t_k \geq 0$ such that

$$Z_k - s_k A - t_k B \succeq 0 \quad \text{for } k = 1, 2, \dots \tag{5}$$

and $Z_k \rightarrow Z$. Then we have

$$z_0^H Z_k z_0 \geq s_k z_0^H A z_0 + t_k z_0^H B z_0 \geq s_k z_0^H A z_0 \geq 0,$$

which implies that $0 \leq s_k \leq z_0^H Z_k z_0 / z_0^H A z_0$ for each k . That is, the sequence $\{s_k\}$ is bounded and hence has a cluster point, say $s_0 \geq 0$. In a similar way, one can prove that the subsequence $\{t_k\}$ has a cluster point, say $t_0 \geq 0$. By (5) we have

$$Z - s_0 A - t_0 B \succeq 0.$$

That is, $Z \in \{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\}$, and so

$$\mathcal{C}_+(D) = \text{cl}\{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\} = \{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

The desired result is proven. \square

Theorem 3.2 can be further generalized. Consider $\triangleright \in \{\geq, \leq, =, \emptyset\}$, where \emptyset means the relation to be unrelated, and denote

$$\triangleright^* \text{ to be } \begin{cases} \geq, & \text{if } \triangleright \text{ is } \geq; \\ \leq, & \text{if } \triangleright \text{ is } \leq; \\ \emptyset, & \text{if } \triangleright \text{ is } =; \\ =, & \text{if } \triangleright \text{ is } \emptyset. \end{cases}$$

THEOREM 3.3. Suppose that $A, B \in \mathcal{H}^n$. Let $D = \{z \in \mathbf{C}^n: z^H A z \geq 0, z^H B z \geq 0\}$. Then

$$\begin{aligned} \{Z \in \mathcal{H}^n: z^H Z z \geq 0, \forall z \in D\} &= (\{Z \geq 0: A \cdot Z \geq 0, B \cdot Z \geq 0\})^* \\ &= \text{cl}\{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 A - \lambda_2 B \geq 0\}. \end{aligned}$$

COROLLARY 3.1. Suppose that $A, B \in \mathcal{H}^n$. Let $D = \{z \in \mathbf{C}^n: z^H A z \geq 0, z^H B z = 0\}$. Suppose furthermore that there are $z_1, z_2, z_3 \in \mathbf{C}^n$ such that $z_1^H A z_1 > 0, z_1^H B z_1 = 0, z_2^H B z_2 > 0$, and $z_3^H B z_3 < 0$. Then

$$\{Z \in \mathcal{H}^n: z^H Z z \geq 0, \forall z \in D\} = \{Z: \exists \lambda_1 \geq 0, \lambda_2 \in \Re, Z - \lambda_1 A - \lambda_2 B \geq 0\}.$$

COROLLARY 3.2. Suppose that $A, B \in \mathcal{H}^n$. Let $D = \{z \in \mathbf{C}^n: z^H A z = 0, z^H B z = 0\}$. Suppose further that there are $z_1, z_2, z_3, z_4 \in \mathbf{C}^n$ such that $z_1^H A z_1 > 0, z_1^H B z_1 = 0, z_2^H A z_2 < 0, z_2^H B z_2 = 0, z_3^H B z_3 > 0$ and $z_4^H B z_4 < 0$. Then

$$\{Z \in \mathcal{H}^n: z^H Z z \geq 0, \forall z \in D\} = \{Z: \exists \lambda_1 \in \Re, \lambda_2 \in \Re, \text{ such that } Z - \lambda_1 A - \lambda_2 B \geq 0\}.$$

Let us now consider nonhomogeneous quadratic functions. Suppose that

$$D = \{z \in \mathbf{C}^n: q_1(z) \geq 0, q_2(z) \geq 0\}, \quad (6)$$

where $q_j(z) = z^H A_j z + 2 \text{Re}(b_j^H z) + c_j, j = 1, 2$.

LEMMA 3.1. Let D be a nonempty set as defined by (6). Then

$$\bigcup_{u \in \mathbf{C}, |u|=1} (u\mathcal{H}(D)) = \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbf{C}^{n+1}: \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix} \cdot \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} \geq 0, j = 1, 2 \right\}.$$

PROOF. That

$$\bigcup_{u \in \mathbf{C}, |u|=1} (u\mathcal{H}(D)) \subseteq \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbf{C}^{n+1}: \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix} \cdot \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} \geq 0, j = 1, 2 \right\}$$

is readily seen by definition. We need only show the other containing relationship. Take any arbitrary

$$\begin{bmatrix} t \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbf{C}^{n+1}: \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix} \cdot \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} \geq 0, j = 1, 2 \right\}. \quad (7)$$

If $t \neq 0$, then

$$\frac{z^H}{\bar{t}} A_j \frac{z}{t} + 2 \text{Re}\left(b_j^H \frac{z}{t}\right) + c_j \geq 0, \quad j = 1, 2.$$

That is, $z/t \in D$. Let $t = \gamma e^{i\alpha}$. Then $[\gamma z^T / e^{i\alpha}]^T \in \mathcal{H}(D)$, which means $[\gamma e^{i\alpha} z^T]^T \in e^{i\alpha}(\mathcal{H}(D))$, implying that $[t z^T]^T \in \bigcup_{u \in \mathbf{C}, |u|=1} (u\mathcal{H}(D))$.

If $t = 0$, then (7) implies that $z^H A_j z \geq 0$ for $j = 1, 2$. In that case, if $z = 0$ then $[0 \ 0^T]^T = [t z^T]^T \in \mathcal{H}(D)$, since $\mathcal{H}(D)$ is a closed cone.

Now we consider the case where $z \neq 0$. Let $z_0 \in D$, i.e., $q_j(z_0) \geq 0$ for $j = 1, 2$. For a given nonzero $\epsilon \in \mathbf{C}$, we consider

$$q_j(z_0 + \epsilon z) = |\epsilon|^2 z^H A_j z + 2 \text{Re}(\epsilon(A_j z_0 + b_j)^H z) + q_j(z_0), \quad j = 1, 2.$$

In order to show that $[0 \ z^T]^T \in \bigcup_{u \in \mathbf{C}, |u|=1} (u\mathcal{H}(D))$, we discuss the following four cases:

Case 1. $\text{Re}(\epsilon(A_j z_0 + b_j)^H z) \geq 0$ for $j = 1, 2$. It follows that $q_j(z_0 + \epsilon z) \geq 0$ for $j = 1, 2$, i.e., $z_0 + \epsilon z \in D$. Let $\epsilon = |\epsilon| e^{i\alpha}$. Then

$$\begin{aligned} z_0 + \epsilon z &= \frac{1/|\epsilon| z_0 + z e^{i\alpha}}{1/|\epsilon|} \in D \\ &\implies \begin{bmatrix} 1/|\epsilon| \\ 1/|\epsilon| z_0 + z e^{i\alpha} \end{bmatrix} \in \mathcal{H}(D) \\ &\implies \begin{bmatrix} 0 \\ z e^{i\alpha} \end{bmatrix} \in \mathcal{H}(D) \\ &\implies \begin{bmatrix} 0 \\ z \end{bmatrix} \in e^{-i\alpha} \mathcal{H}(D) \subseteq \bigcup_{u \in \mathbf{C}, |u|=1} (u\mathcal{H}(D)). \end{aligned}$$

Case 2. $\text{Re}(\epsilon(A_j z_0 + b_j)^H z) \leq 0$ for $j = 1, 2$. Replacing ϵ by $-\epsilon$, this reduces to the previous case.

Case 3. $\text{Re}(\epsilon(A_1 z_0 + b_1)^H z) < 0$ and $\text{Re}(\epsilon(A_2 z_0 + b_2)^H z) > 0$. Let $(A_j z_0 + b_j)^H z = \gamma_j e^{i\beta_j}$ for $j = 1, 2$. Then in this case we have $\text{Re}(|\epsilon| \gamma_1 e^{i(\alpha+\beta_1)}) < 0$ and $\text{Re}(|\epsilon| \gamma_2 e^{i(\alpha+\beta_2)}) > 0$, i.e., $\text{Re} e^{i(\alpha+\beta_1)} < 0$ and $\text{Re} e^{i(\alpha+\beta_2)} > 0$. By rotating $e^{i(\alpha+\beta_1)}$ and $e^{i(\alpha+\beta_2)}$ simultaneously, it follows that there is $\Delta\alpha$ such that $\text{Re} e^{i(\alpha+\Delta\alpha+\beta_1)} \geq 0$ and $\text{Re} e^{i(\alpha+\Delta\alpha+\beta_2)} \geq 0$. Therefore we have $\text{Re}(|\epsilon| \gamma_1 e^{i(\alpha+\Delta\alpha+\beta_1)}) \geq 0$ and $\text{Re}(|\epsilon| \gamma_2 e^{i(\alpha+\Delta\alpha+\beta_2)}) \geq 0$. This implies that $q_j(z_0 + |\epsilon| e^{i(\alpha+\Delta\alpha)} z) \geq 0$, $j = 1, 2$, i.e., $z_0 + |\epsilon| e^{i(\alpha+\Delta\alpha)} z \in D$. Then using a similar argument as in Case 1, we conclude that

$$\begin{bmatrix} 0 \\ z \end{bmatrix} \in (e^{-i(\alpha+\Delta\alpha)}) \mathcal{H}(D) \subseteq \bigcup_{u \in \mathbb{C}, |u|=1} (u \mathcal{H}(D)).$$

Case 4. $\text{Re}(\epsilon(A_1 z_0 + b_1)^H z) > 0$ and $\text{Re}(\epsilon(A_2 z_0 + b_2)^H z) < 0$. Replacing ϵ by $-\epsilon$, this is the same as Case 3. The proof is thus complete. \square

We remark that Lemma 3.1 is an extension of Sturm and Zhang [9, Lemma 2], where D is defined by only one real quadratic function.

THEOREM 3.4. Let $D = \{z \in \mathbb{C}^n: q_j(z) = z^H A_j z + 2 \text{Re}(b_j^H z) + c_j \geq 0, j = 1, 2\}$ be nonempty. Suppose that there is $z_0 \in \mathbb{C}^n$ such that $q_1(z_0) > 0$ and $q_2(z_0) > 0$. Then

$$\mathcal{F}\mathcal{C}_+(D) = \{Z: \exists \lambda_1 \geq 0, \lambda_2 \geq 0 \text{ such that } Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \geq 0\}.$$

PROOF. It follows that

$$\begin{aligned} \mathcal{F}\mathcal{C}_+(D) &= \mathcal{C}_+ \left(\bigcup_{u \in \mathbb{C}, |u|=1} (u \mathcal{H}(D)) \right) \quad (\text{Proposition 3.2}) \\ &= \left(\text{cone} \left\{ \begin{bmatrix} t \\ z \end{bmatrix} [\bar{t}, z^H]: \begin{bmatrix} t \\ z \end{bmatrix} \in \bigcup_{u \in \mathbb{C}, |u|=1} (u \mathcal{H}(D)) \right\} \right)^* \quad (\text{Proposition 3.1}) \\ &= \left(\text{conv} \left\{ \begin{bmatrix} t \\ z \end{bmatrix} [\bar{t}, z^H]: \begin{bmatrix} t \\ z \end{bmatrix} \in \bigcup_{u \in \mathbb{C}, |u|=1} (u \mathcal{H}(D)) \right\} \right)^* \quad (\text{Theorem 3.1}) \\ &= \left(\text{conv} \left\{ \begin{bmatrix} t \\ z \end{bmatrix} [\bar{t}, z^H]: \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix} \cdot \begin{bmatrix} t \\ z \end{bmatrix} [\bar{t}, z^H] \geq 0, j = 1, 2 \right\} \right)^* \quad (\text{Lemma 3.1}) \\ &= (\{Z \in \mathcal{H}_+^{n+1}: M(q_j(\cdot)) \cdot Z \geq 0, j = 1, 2\})^* \\ &= \text{cl}\{Z \in \mathcal{H}_+^{n+1}: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \geq 0\} \\ &= \{Z \in \mathcal{H}_+^{n+1}: \exists \lambda_1 \geq 0, \lambda_2 \geq 0, Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \geq 0\}. \quad \square \end{aligned}$$

We remark that Theorem 3.4 can be regarded as a generalization of Sturm and Zhang [9, Corollary 5]. Note that from Proposition 3.2 we have $\mathcal{F}\mathcal{C}_+(\mathbb{C}^n) = \mathcal{H}_+^{n+1}$. Then Theorem 3.4 implies that whenever $D = \{z: q_j(z) \geq 0, j = 1, 2\}$ has a strict interior point, then $q(z) \geq 0$ for all $z \in D$ if and only if there are $\lambda_1, \lambda_2 \geq 0$ such that $q(z) - \lambda_1 q_1(z) - \lambda_2 q_2(z) \geq 0$ for $z \in \mathbb{C}^n$.

4. Complex quadratic programming and SDP relaxation. Consider the complex quadratically constrained quadratic programming:

$$\begin{aligned} (\text{QCQP}) \quad & \max \quad z^H Q z + 2 \text{Re} z^H q \\ & \text{s.t.} \quad z^H A_j z + 2 \text{Re} z^H b_j + c_j \leq 0, \quad j = 1, \dots, m, \end{aligned}$$

where $Q, A_j \in \mathcal{H}^n$, $b_j, q \in \mathbb{C}^n$, $c_j \in \mathbb{R}$, $j = 1, \dots, m$. Denote

$$B_0 = \begin{bmatrix} 0 & q^H \\ q & Q \end{bmatrix}, \quad B_j = \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix}, \quad j = 1, \dots, m, \quad B_{m+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We rewrite (QCQP) equivalently as

$$\begin{aligned} (\text{QCQP}) \quad & \max \quad B_0 \cdot \begin{bmatrix} 1 & z^H \\ z & z z^H \end{bmatrix} \\ & \text{s.t.} \quad B_j \cdot \begin{bmatrix} 1 & z^H \\ z & z z^H \end{bmatrix} \leq 0, \quad j = 1, \dots, m. \end{aligned}$$

A homogenized version of (QCQP) is

$$\begin{aligned}
 \text{(HQ)} \quad & \max B_0 \bullet \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} \\
 \text{s.t.} \quad & B_j \bullet \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} \leq 0, \quad j = 1, \dots, m, \\
 & B_{m+1} \bullet \begin{bmatrix} |t|^2 & tz^H \\ \bar{t}z & zz^H \end{bmatrix} = 1.
 \end{aligned}$$

It follows that if $[t \ z^T]^T$ solves (HQ), then z/t solves (QCQP). The SDP relaxation of (QCQP) is

$$\begin{aligned}
 \text{(QCQPR)} \quad & \max B_0 \bullet Z \\
 \text{s.t.} \quad & B_j \bullet Z \leq 0, \quad j = 1, \dots, m, \\
 & B_{m+1} \bullet Z = 1, \\
 & Z \geq 0,
 \end{aligned}$$

with the dual given by

$$\begin{aligned}
 \text{(DQCQPR)} \quad & \min y_0 \\
 \text{s.t.} \quad & Y = \sum_{j=1}^m y_j B_j - B_0 + y_0 B_{m+1} \geq 0, \\
 & y_j \geq 0, \quad j = 1, \dots, m, \quad y_0 \text{ free.}
 \end{aligned}$$

Throughout this section, we assume that (QCQP) satisfies the Slater condition, i.e., there exists $z_0 \in \mathbf{C}^n$ such that $q_j(z_0) < 0$, $j = 1, \dots, m$. Accordingly, (QCQPR) satisfies the Slater condition as well.

It is well known that (QCQP) is NP-hard in general. In the remainder of this section, we shall study (QCQP) where m is small.

THEOREM 4.1. *Suppose that (QCQPR) and (DQCQPR) have complementary optimal solutions. Moreover, suppose that $m = 2$. Then (QCQP) and (QCQPR) have the same optimal value. Moreover, an optimal solution for (QCQP) can be constructed from an optimal solution for (QCQPR) in polynomial time.*

PROOF. Let $Z^* \geq 0$ be an optimal solution of (QCQPR), and $(y_0^*, y_1^*, y_2^*, Y^*)$ be an optimal solution of (DQCQPR). Since the strong duality is satisfied, the respective primal and dual optimal solutions are complementary, i.e., $Z^* \bullet Y^* = 0$.

Denote “ \leq ” to be either “ $<$ ” or “ $=$ ”. Then $B_j \bullet Z^* \leq_j 0$, $j = 1, 2$. By Theorem 2.1 (or Corollary 2.1), there exist nonzero z_k , $k = 1, \dots, r$, where r is the rank of Z^* such that

$$Z^* = \sum_{k=1}^r z_k z_k^H, \quad B_j \bullet z_k z_k^H \leq_j 0, \quad j = 1, 2, \quad k = 1, \dots, r.$$

Since $Z_{11}^* = 1$, there is $l \in \{1, \dots, r\}$ such that $t_l \neq 0$ where $z_l = [t_l \ \bar{z}_l^T]^T$. Therefore, $q_j(\bar{z}_l/t_l) \leq_j 0$, $j = 1, 2$, which implies that \bar{z}_l/t_l is a feasible solution for (QCQP).

Due to $Y^* \geq 0$, we have $Y^* \bullet z_l z_l^H = 0$. If $B_j \bullet Z^* < 0$, then by complementarity it follows that $y_j^* = 0$. If $B_j \bullet Z^* = 0$, then by the decomposition theorem we have $B_j \bullet z_l z_l^H = 0$. Therefore, we always have $y_j^*(B_j \bullet z_l z_l^H) = 0$, $j = 1, 2$. This, combining with $Y^* \bullet z_l z_l^H = 0$, leads to the conclusion that $[1 \ \bar{z}_l^T/t_l]^T [1 \ \bar{z}_l^H/\bar{t}_l]$ is a solution of (QCQPR). Therefore \bar{z}_l/t_l is optimal for (QCQP). Note that the rank-one decomposition procedure runs in polynomial time. The theorem is proven. \square

We remark that Theorem 4.1 is an extension of Ye and Zhang [11, Theorem 2.3]. Note further that we do not have to assume a nonbinding condition in Theorem 4.1, which however is necessary in Ye and Zhang [11].

Not all SDP problems are relaxations of quadratic programs. In the next section, we shall study the rank of optimal solutions for a general complex SDP problem.

5. The low rank optimal solutions for standard complex SDP. Consider a standard (complex) SDP problem

$$\begin{aligned} \text{(SDP)} \quad & \min C \bullet Z \\ & \text{s.t. } A_j \bullet Z = b_j, \quad j = 1, \dots, m, \\ & Z \succeq 0, \end{aligned}$$

and its dual

$$\begin{aligned} \text{(DSDP)} \quad & \max \sum_{j=1}^m y_j b_j \\ & \text{s.t. } \sum_{j=1}^m y_j A_j \preceq C. \end{aligned}$$

Suppose that (SDP) and (DSDP) have a complementary optimal solution pair.

Consider now an optimal solution for (SDP) with minimum rank. In particular, let S_p be the set of all optimal solutions for (SDP), introduce

$$r_p = \min\{\text{rank}(Z) : Z \in S_p\},$$

and let Z^* be such an optimal solution with $\text{rank}(Z^*) = r_p$.

For this given Z^* , let us introduce the following notion of minimum diagonal rank:

$$r_M = \min\{\text{rank}(\text{diag}(V^H A_1 V), \dots, \text{diag}(V^H A_m V)) : VV^H = Z^*, \text{rank}(V) = \text{rank}(Z^*) = r_p\}. \quad (8)$$

The result below follows from Theorem 2.1.

PROPOSITION 5.1. *Suppose that $S_p \neq \emptyset$ and $m \geq 3$. We have $r_M \leq m - 2$.*

PROOF. Consider the case that at least three of the values $\{b_1, b_2, \dots, b_m\}$ are nonzero. (In the other case, namely at most two of the b_j 's are nonzero, the proof can be easily adapted.) Without losing generality, let us assume that b_1, b_2 , and b_3 are nonzero.

Take any $Z \in S_p$ with rank r . Since

$$(A_1/b_1 - A_2/b_2) \bullet Z = 0 \quad \text{and} \quad (A_1/b_1 - A_3/b_3) \bullet Z = 0,$$

Theorem 2.1 asserts that there is a rank-one decomposition of Z ,

$$Z = \sum_{k=1}^r v_k v_k^H,$$

such that

$$v_k^H (A_1/b_1 - A_2/b_2) v_k = 0, \quad \text{and} \quad v_k^H (A_1/b_1 - A_3/b_3) v_k = 0$$

for all $k = 1, \dots, r$. Letting $V = [v_1, \dots, v_r]$, we thus have

$$\text{diag}(V^H A_1 V)/b_1 = \text{diag}(V^H A_2 V)/b_2 = \text{diag}(V^H A_3 V)/b_3.$$

Hence,

$$\text{rank}(\text{diag}(V^H A_1 V), \dots, \text{diag}(V^H A_m V)) \leq m - 2.$$

The proposition is proven. \square

Let us denote d_D to be the dimension of the optimal solution set for (DSDP). Theorem 5.1 gives an upper bound on the lowest rank among all the optimal solutions.

THEOREM 5.1. *Suppose that the complex SDP pair (SDP) and (DSDP) have a complementary optimal solution pair, and $m \geq 3$. Then, it holds that*

$$r_p \leq \min\{r_M, \lfloor \sqrt{m - d_D} \rfloor\}.$$

PROOF. For $Z^* \in S_p$ with $\text{rank}(Z^*) = r_p$, let $Z^* = VV^H$ be the decomposition attaining the minimum diagonal rank as defined in (8). Consider the following system of linear equations:

$$\begin{aligned} (\text{diag}(V^H A_1 V))^T x &= A_1 \cdot \left(\sum_{k=1}^{r_p} x_k v_k v_k^H \right) = 0 \\ (\text{diag}(V^H A_2 V))^T x &= A_2 \cdot \left(\sum_{k=1}^{r_p} x_k v_k v_k^H \right) = 0 \\ &\vdots \\ (\text{diag}(V^H A_m V))^T x &= A_m \cdot \left(\sum_{k=1}^{r_p} x_k v_k v_k^H \right) = 0. \end{aligned}$$

The rank of the coefficient matrix of the above linear equation is r_M and the number of (real) variables is r_p ($x \in \mathbb{R}^{r_p}$). If $r_p > r_M$, then the above equation must have a nonzero solution $x \in \mathbb{R}^{r_p}$ with $x \not\leq 0$. In that case, it would follow that

$$Z^*(t) = \sum_{k=1}^{r_p} (1 - tx_k) v_k v_k^H$$

is also an optimal solution for (SDP) for all $t \leq \hat{t} := \min\{1/x_k : x_k > 0, k = 1, \dots, r_p\}$. In particular, $\text{rank}(Z^*(\hat{t})) \leq r_p - 1$, contradicting the fact that Z^* is the minimum rank optimal solution. This shows that $r_p \leq r_M$.

We do not necessarily restrict the directions to be diagonal. As an alternative, consider the equation

$$\left. \begin{aligned} (V^H A_1 V) \cdot \Delta &= 0 \\ (V^H A_2 V) \cdot \Delta &= 0 \\ &\vdots \\ (V^H A_m V) \cdot \Delta &= 0 \end{aligned} \right\}, \tag{9}$$

where $\Delta \in \mathcal{H}^{r_p}$.

Due to the complementarity, we have

$$\left(C - \sum_{j=1}^m y_j A_j \right) V V^H = 0$$

for any dual optimal solution (y_1, \dots, y_m) . Thus, by the positive semidefiniteness of the matrices, we have

$$V^H C V - \sum_{j=1}^m y_j V^H A_j V = 0.$$

Therefore, the rank of the coefficient matrix in (9) is $m - d_D$. Since the dimension of Δ is r_p^2 , this implies that as long as $r_p^2 > m - d_D$, Equation (9) would admit a nonzero solution $\Delta \in \mathcal{H}^{r_p}$, allowing $V(I - t\Delta)V^H$ to be optimal for (SDP) with t satisfying $t\Delta \leq I$, thereby enabling a possibility to further reduce the rank of Z^* . Again, this contradicts the fact that Z^* has minimum rank among optimal solutions. This in turn shows that we must have $r_p^2 \leq m - d_D$. Since r_p is integer, we thus have $r_p \leq \lfloor \sqrt{m - d_D} \rfloor$. The theorem is proven. \square

Note that since the diagonal of a matrix is only a part of the whole matrix, naturally we have $r_M \leq m - d_D$. In general, we may expect r_M to be much less than $m - d_D$ indeed.

As a consequence of Theorem 5.1, we conclude that (SDP) has a rank-one optimal solution if $m \leq 3$, since in this case it follows from Proposition 5.1 and Theorem 5.1 that $r_p \leq \min\{3 - 2, \lfloor \sqrt{3 - 0} \rfloor\} = 1$. Hence Theorem 5.1 generalizes a result in Ye and Zhang [11, §2.2].

It also follows from Theorem 5.1 that if $m = 4$ and the dual optimal solution is not unique ($d_D \geq 1$), then there is a rank-one optimal solution for (SDP).

Remark that in the real case the discussion on the minimum rank optimal solutions for SDP can be found in Barvinok [2], Pataki [6], Ye and Zhang [11], and Ye [10].

6. Joint numerical range and matrix decomposition. Let $A_j \in \mathcal{H}^n$ for $j = 1, \dots, m$ be Hermitian matrices, and $D \in \mathcal{H}_{++}^n$ be a Hermitian positive definite matrix. The joint numerical range of this family $\{A_1, \dots, A_m, D\}$ of Hermitian matrices is defined as

$$\mathcal{F}(A_1, \dots, A_m; D) := \left\{ \begin{pmatrix} A_1 \cdot zz^H \\ \vdots \\ A_m \cdot zz^H \end{pmatrix} : D \cdot zz^H = 1, z \in \mathbf{C}^n \right\}.$$

To proceed, let us introduce the following set:

$$\mathcal{M}(A_1, \dots, A_m; D) := \left\{ \begin{pmatrix} A_1 \cdot Z \\ \vdots \\ A_m \cdot Z \end{pmatrix} : D \cdot Z = 1, Z \in \mathcal{H}_+^n \right\}.$$

Furthermore, we define the set

$$\mathcal{F}(A_1, \dots, A_m) := \left\{ \begin{pmatrix} A_1 \cdot zz^H \\ \vdots \\ A_m \cdot zz^H \end{pmatrix} : z \in \mathbf{C}^n \right\},$$

and similarly the set $\mathcal{M}(A_1, \dots, A_m)$.

It is elementary to check that $\mathcal{M}(A_1, \dots, A_m; D)$ is the convex hull of $\mathcal{F}(A_1, \dots, A_m; D)$, hence convex. Moreover, $\mathcal{F}(A_1, \dots, A_m)$ and $\mathcal{M}(A_1, \dots, A_m)$ are cones, and so $\mathcal{M}(A_1, \dots, A_m)$ is in fact a convex cone in \Re^m .

We shall see below that a certain rank restriction on Z in fact does not affect the numerical range $\mathcal{M}(A_1, \dots, A_m; D)$, by applying the decomposition theorem discussed in §2.

PROPOSITION 6.1. *Let $\{A_1, \dots, A_m; D\}$ be Hermitian matrices with $D > 0$. Suppose that $m \geq 2$. It holds that*

$$\mathcal{M}(A_1, \dots, A_m; D) := \left\{ \begin{pmatrix} A_1 \cdot Z \\ \vdots \\ A_m \cdot Z \end{pmatrix} : D \cdot Z = 1, Z \in \mathcal{H}_+^n, \text{rank}(Z) \leq m - 1 \right\}.$$

PROOF. Take an arbitrary $v \in \mathcal{M}(A_1, \dots, A_m; D)$. Let $Z \in \mathcal{H}_+^n$ be such that

$$A_i \cdot Z = v_i, \quad i = 1, \dots, m, \quad D \cdot Z = 1.$$

Obviously, we have

$$(A_{m-1} - v_{m-1}D) \cdot Z = 0, \quad \text{and} \quad (A_m - v_m D) \cdot Z = 0.$$

According to Theorem 2.1, there is a particular rank-one decomposition of Z , say $Z = \sum_{j=1}^r z_j z_j^H$ with $r = \text{rank } Z$, such that

$$(A_{m-1} - v_{m-1}D) \cdot z_j z_j^H = 0, \quad (A_m - v_m D) \cdot z_j z_j^H = 0, \quad j = 1, \dots, r.$$

Let us define an $(m - 1) \times r$ matrix $B = (b_{ij})$ with entries given by

$$b_{ij} := (A_i - v_i D) \cdot z_j z_j^H, \quad i = 1, \dots, m - 2, \quad j = 1, \dots, r, \\ b_{m-1,j} := D \cdot z_j z_j^H, \quad j = 1, \dots, r,$$

and consider the following equation:

$$Bt = e_{m-1}, \quad t \geq 0, \tag{10}$$

where e_{m-1} is the $(m - 1)$ -dimensional unit vector whose last component is 1 and 0 elsewhere. Clearly, by the construction of the matrix B , we have $Be = e_{m-1}$ with e being the vector of all ones. Therefore, there must exist some $\hat{t} \geq 0$ satisfying (10) such that the number of its components does not exceed $m - 1$, which is the number of rows in B . Let

$$\hat{Z} = \sum_{j=1}^r \hat{t}_j z_j z_j^H.$$

Hence we have

$$D \cdot \hat{Z} = 1, \quad v = (A_1 \cdot \hat{Z}, \dots, A_{m-1} \cdot \hat{Z}, A_m \cdot \hat{Z})^T, \quad \text{and} \quad \text{rank}(\hat{Z}) \leq m - 1.$$

This completes the proof. \square

By the same argument, one has the following:

PROPOSITION 6.2. *Let $\{A_1, \dots, A_m\}$ be Hermitian matrices, and $m \geq 3$. It holds that*

$$\mathcal{M}(A_1, \dots, A_m) := \left\{ \begin{pmatrix} A_1 \cdot Z \\ \vdots \\ A_m \cdot Z \end{pmatrix} : Z \in \mathcal{H}_+^n, \text{rank}(Z) \leq m - 2 \right\}. \quad (11)$$

Interestingly, Proposition 6.1 implies that $\mathcal{M}(A_1, A_2; D) = \mathcal{F}(A_1, A_2; D)$, or equivalently that the set

$$\left\{ \begin{pmatrix} z^H A_1 z \\ z^H A_2 z \end{pmatrix} : z^H z = 1, z \in \mathbf{C}^n \right\}$$

is convex. The last statement is a well-known result of Hausdorff [5].

Similarly, a consequence of Proposition 6.2 ($m = 3$) is that $\mathcal{M}(A_1, A_2, A_3) = \mathcal{F}(A_1, A_2, A_3)$, and so

$$\left\{ \begin{pmatrix} z^H A_1 z \\ z^H A_2 z \\ z^H A_3 z \end{pmatrix} : z \in \mathbf{C}^n \right\}$$

is a convex cone in \mathfrak{R}^3 , which was first established by Brickman [3]. Furthermore, Polyak [8] was able to characterize when this cone is pointed, under an additional condition that $n \geq 3$. Results like Hausdorff's theorem and Brickman's theorem as shown above are important; e.g., Brickman's theorem can be directly applied to prove the S-Lemma, Theorem 3.2, by a separation argument. Historically, in their approach to establish the S-Lemma in the Hermitian case, Fradkov and Jakubovich [4] first proved Brickman's theorem (obviously not being aware of Brickman [3]), and then applied the separation argument. Much information on the S-Lemma and related topics can be found in a recent survey by Pólik and Terlaky [7]. It is clear that both Hausdorff's and Brickman's theorems can be proved in very different ways (see, e.g. Au-Yeung and Poon [1] for a quite unexpected derivation). Understandably, since there is "no free lunch," all proof techniques would require some efforts and be powerful enough to claim the strong results such as Hausdorff's and Brickman's theorems. Our approach as described in this section, therefore, offers one of the alternative paths to this destination, in our case via the rank-one decomposition of the Hermitian positive semidefinite matrices established in §2.

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