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Generalized Constraint Qualifications and Optimality Conditions for Set-Valued Optimization Problems

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In this paper we discuss the connections of four generalized constraint qualifications for set-valued vector optimization problems with constraints. Then some K-T type necessary and sufficient optimality conditions are derived, in terms of the contingent epiderivatives. © 2002 Elsevier Science

Key Words: set-valued maps; generalized constraint qualifications; optimality conditions; contingent epiderivatives.

1. INTRODUCTION

In recent years, there has been an increasing amount of attention to optimality conditions for set-valued optimization problems. For instance, Jahn and Rauh [1] introduced the notion of the contingent epiderivative of a set-valued map and derived the formulation of optimality conditions in the unconstrained set-valued optimization. Li [3] obtained the optimality conditions for set-valued optimization by applying the alternative theorem in real linear space without any topology. Lin [5] defined a weak subdifferential of set-valued maps in real linear topological spaces, and the Moreau–Rockafellar type and the Farkas–Minkowski type theorems for set-valued maps were generalized. Through these, the K-T type necessary conditions for the existence of a weak minimal point have been given. Yang [7] introduced Dini directional derivatives for set-valued maps, in terms of which a derivative concept of a Jacobificator for set-valued maps

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is introduced. Their applications were given to present optimality conditions and mean value theorems.

In this paper we are concerned with the generalized constraint qualifications and the optimality conditions for set-valued optimization problems. In Section 2, we give some notations and preliminaries. In Section 3, we introduce four constraint qualifications and investigate the connections between them. In Section 4, we obtain some Kuhn–Tucker type sufficient and necessary conditions for set-valued optimization problems with constraints, in terms of contingent epiderivatives.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, let X, Y, Z, and W be real normed linear spaces. Let Y, Z, and W be partially ordered by pointed convex cones Y_+, Z_+, W_+, respectively. Assume that the interiors of Y_+, Z_+, denoted by $\operatorname{int} Y_+$, $\operatorname{int} Z_+$ respectively, are both nonempty. However, the interior of W_+ is not necessary to be nonempty. We denote by Y^*, Z^*, W^* the dual spaces of Y, Z, W, respectively. The dual cone Y_+^* of Y_+ is defined by $Y_+^* = \{y^* \in Y^* \mid \langle y, y^* \rangle \geq 0, \forall y \in Y_+\}$, where $\langle y, y^* \rangle$ denotes the value of the linear continuous functional y^* at the point y. The meanings of Z_+^*, W_+^* are similar.

Let D be a nonempty subset of X, let F: $D \to 2^Y$, G: $D \to 2^Z$, H: $D \to 2^W$ be set-valued maps such that $F(x) \neq \emptyset$, $G(x) \neq \emptyset$, $H(x) \neq \emptyset$, $\forall x \in D$. Let

$$F(D) = \bigcup_{x \in D} F(x), \qquad \langle F(x), y^* \rangle = \{\langle y, y^* \rangle \mid y \in F(x)\},$$

$$\langle F(D), y^* \rangle = \bigcup_{x \in D} \langle F(x), y^* \rangle.$$

We denote by R the set of real numbers, by N the set of natural numbers, by O the null element of every space. For $A \subset R$, $b \in R$, write $A \geq$
$(\leq, >, <)b$, iff $a \geq (\leq, >, <)b$, $\forall a \in A$.

DEFINITION 2.1. (a) A subset M in Y is called nearly convex if $\exists \alpha \in (0,1)$, such that for $\forall y_1, y_2 \in M$, $\alpha y_1 + (1-\alpha) y_2 \in M$.

(b) The set-valued map F: $D \to 2^Y$ is Y_+-convex on D if D is convex, and for $x_1, x_2 \in D$ and $\forall \lambda \in (0,1)$, $\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + Y_+$.

(c) The set-valued map F: $D \to 2^Y$ is nearly Y_+-convexlike on D, if there is $\lambda \in (0,1)$ such that for $\forall x_1, x_2 \in D$, $\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(D) + Y_+$.

It is well known that

$$Y_+\text{-convex} \Rightarrow Y_+\text{-convexlike (see [8])} \Rightarrow Y_+\text{-subconvexlike (see [8])} \Rightarrow \text{generalized}$$

Y_+-subconvexlike (see [17]), and that

$$Y_+\text{-convexlike} \Rightarrow \text{nearly } Y_+\text{-convexlike}.$$

DEFINITION 2.2. (a) The set epi $F = \{(x, y) \in X \times Y \mid x \in D,\ y \in F(x) + Y_+\}$ is called the epigraph of F on D.

(b) Let S be a nonempty subset of W, and let $s_0 \in \operatorname{cl} S$ (closure of S) be a given element. A vector $h \in W$ is called a tangent vector to S at s_0, if there are a sequence $(z_n)_{n \in N}$ of elements in S and a sequence $(\lambda_n)_{n \in N}$ of positive real numbers with $s_0 = \lim_{n \to \infty} z_n$ and $h = \lim_{n \to \infty} \lambda_n(z_n - s_0)$. The set $T(S, s_0)$ of all tangent vectors to S at s_0 is called a contingent cone to S at s_0.

(c) Let a pair $(x_0, y_0) \in X \times Y$ with $x_0 \in D$ and $y_0 \in F(x_0)$ be given. Denote by $T_X(\text{epi}(F), (x_0, y_0))$ the projection of $T(\text{epi}(F), (x_0, y_0))$ on X. A single-valued map $DF(x_0, y_0)$: $T_X(\text{epi}(F), (x_0, y_0)) \subset X \to Y$, whose epigraph equals the contingent cone to the epigraph of F at (x_0, y_0), i.e., $\text{epi}(DF(x_0, y_0)) = T(\text{epi}(F), (x_0, y_0))$, is called contingent epiderivative of F at (x_0, y_0).

Consider the set-valued optimization problem (P)

$$\text{(P)} \quad \min F(x) \qquad \text{s.t. } x \in K,$$

where the feasible set $K = \{x \in D \mid -G(x) \cap Z_+ \neq \emptyset,\ \ -H(x) \cap W_+ \neq \emptyset\}$.

DEFINITION 2.3. A point (x_0, y_0, z_0, w_0) is called a feasible point of (P) if $x_0 \in K$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap -Z_+$, $w_0 \in H(x_0) \cap -W_+$. A point $x_0 \in K$ is called a weakly efficient solution of (P), if $\exists y_0 \in F(x_0)$ such that $(y_0 - F(K)) \cap \operatorname{int} Y_+ = \emptyset$. In this case, the point (x_0, y_0, z_0, w_0) is called a weak minimizer of (P).

Remark 2.1. It is obvious that when $W_+ = \{O\}$, (P) can be written as (P′),

$$\text{(P}'\text{)} \quad \min F(x) \qquad \text{s.t. } x \in K',$$

where the set $K' = \{x \in D \mid -G(x) \cap Z_+ \neq \emptyset,\ O \in H(x)\}$. Since (P′) is a particular case of (P), we are concerned in this paper with Problem (P).

The following lemmas will be very useful.

LEMMA 2.1 (see [9]). *If $M \subset Y$ is a nearly convex set, then* int M *is a convex set.*

LEMMA 2.2 (see [6]). *If $y_0^* \in Y_+^* \backslash \{O\}$, $y_0 \in \operatorname{int} Y_+$, then $\langle y_0, y_0^* \rangle > 0$.*

LEMMA 2.3 (see [9]). *Let $M \subset Y$ be a nearly convex set, and let* $\operatorname{int} M \neq \varnothing$. *Suppose that for each $y^* \in Y^* \backslash \{O\}$, $\langle u, y^* \rangle > 0$, $\forall u \in \operatorname{int} M$. Then $\langle u, y^* \rangle \geq 0$, $\forall u \in M$.*

3. GENERALIZED CONSTRAINT QUALIFICATIONS

In what follows, we put $U = Z \times W$, $U_+ = Z_+ \times W_+$, $J = (G, H)\colon D \to 2^U$. The notation $(G, H)(x)$ is used for $G(x) \times H(x)$ here. One can easily verify that $U^* = (Z \times W)^* = Z^* \times W^*$, and that $U_+^* = (Z_+ \times W_+)^* = Z_+^* \times W_+^*$.

Now we consider the following four generalized constraint qualifications (Q_0) to (Q_3). Let $R_- = \{r \in R \mid r < 0\}$.

(Q_0) $\exists x' \in D$ such that $-G(x') \cap \operatorname{int} Z_+ \neq \varnothing$, $-\operatorname{int} H(x') \cap W_+ \neq \varnothing$.

(Q_1) (i) $\exists x' \in D$ such that $-G(x') \cap \operatorname{int} Z_+ \neq \varnothing$, $-H(x') \cap W_+ \neq \varnothing$.

(ii) $-\operatorname{int} H(D) \cap W_+ \neq \varnothing$.

(Q_2) (i) For each $z^* \in Z_+^* \backslash \{O\}$, there is $x' \in D$ such that $\langle G(x'), z^* \rangle \cap R_- \neq \varnothing$, $-H(x') \cap W_+ \neq \varnothing$.

(ii) For each $w^* \in W_+^* \backslash \{O\}$, there is $x' \in D$ such that $\langle H(x'), w^* \rangle \cap R_- \neq \varnothing$.

(Q_3) For each $(z^*, w^*) \in Z_+^* \times W_+^* \backslash \{(O, O)\}$, there is $x' \in D$ such that

$$(\langle G(x'), z^* \rangle + \langle H(x'), w^* \rangle) \cap R_- \neq \varnothing.$$

THEOREM 3.1. *The following implications are true*:

$$(Q_0) \Rightarrow (Q_1) \Rightarrow (Q_2) \Rightarrow (Q_3).$$

Proof. It is easy to show $(Q_0) \Rightarrow (Q_1)$. Now we prove $(Q_1) \Rightarrow (Q_2)$. Suppose that (Q_1) holds.

(a) Let $z^* \in Z_+^* \backslash \{O\}$. Hence $\exists x' \in D$ such that $-G(x') \cap \operatorname{int} Z_+ \neq \varnothing$. By Lemma 2.2 we have $\langle \operatorname{int} Z_+, z^* \rangle > 0$. Therefore $\langle G(x'), z^* \rangle \cap R_- \neq \varnothing$.

(b) Let $w^* \in W_+^* \backslash \{O\}$. Thus $\exists w_0 \in W$ such that $\langle w_0, w^* \rangle > 0$. Let $w_1 \in -\operatorname{int} H(D) \cap W_+$. So, $\exists \lambda > 0$ such that $w_1 + \lambda w_0 \in -\operatorname{int} H(D) \subset -H(D)$. Hence $\exists x' \in D$ such that $w_1 + \lambda w_0 \in -H(x')$. Since $w_1 \in W_+$, then $\langle w_1 + \lambda w_0, w^* \rangle = \langle w_1, w^* \rangle + \langle \lambda w_0, w^* \rangle > 0$. Therefore $\langle H(x'), w^* \rangle \cap R_- \neq \emptyset$.

Next we show $(Q_2) \Rightarrow (Q_3)$. Assume that (Q_2) holds. Let $(z^*, w^*) \in Z_+^* \times W_+^* \backslash \{(O, O)\}$.

If $z^* = O$, then (Q_3) is introduced directly by (ii) of (Q_2). If $z^* \neq O$, then from (i) of (Q_2), $\exists x' \in D$, such that $\langle G(x'), z^* \rangle \cap R_- \neq \emptyset$, $-H(x') \cap W_+ \neq \emptyset$. Thus $\langle H(x'), w^* \rangle \cap (R_- \cup \{0\}) \neq \emptyset$. Hence $\exists r_1 \in \langle G(x'), z^* \rangle \cap R_-$, $r_2 \in \langle H(x'), w^* \rangle \cap (R_- \cup \{0\})$ such that $r_1 + r_2 < 0$. This means $(\langle G(x'), z^* \rangle + \langle H(x'), w^* \rangle) \cap R_- \neq \emptyset$. ∎

LEMMA 3.1. $\operatorname{int}(J(D) + U_+) \neq \emptyset$, *if and only if* $\operatorname{int}(J(D) + (\operatorname{int} Z_+) \times W_+) \neq \emptyset$.

Indeed, the proof of the last lemma is similar to the proof of Lemma 2.2 [9], when J is a vector-valued map.

LEMMA 3.2. *If* $u^* = (z^*, w^*) \in U_+^* = Z_+^* \times W_+^*$, $z^* \neq O$, $u = (z, w) \in (\operatorname{int} Z_+) \times W_+$, *then* $\langle u, u^* \rangle > 0$.

Proof. Since $z^* \in Z_+^* \backslash \{O\}$, and since $z \in \operatorname{int} Z_+$, thus $\langle z, z^* \rangle > 0$. Hence $\langle u, u^* \rangle = \langle z, z^* \rangle + \langle w, w^* \rangle > 0$, because of $\langle w, w^* \rangle \geq 0$. ∎

LEMMA 3.3. *If the set-valued map* J: $D \to 2^U$ *is nearly* U_+*-convexlike on* D, *then* $M = J(D) + (\operatorname{int} Z_+) \times W_+$ *is nearly convex.*

Proof. Let $m_1, m_2 \in M$; then $\exists x_i \in D$, $u_i \in (\operatorname{int} Z_+) \times W_+$, such that $m_i \in J(x_i) + u_i$, $i = 1, 2$. Since J is nearly U_+-convexlike, there exists $\alpha \in (0, 1)$; for the previous $x_1, x_2 \in D$, we have

$$\alpha J(x_1) + (1 - \alpha) J(x_2) \subset J(D) + U_+.$$

Thus $\alpha(m_1 - u_1) + (1 - \alpha)(m_2 - u_2) \in J(D) + U_+$. Because the set $(\operatorname{int} Z_+) \times W_+$ is convex, so we have $u_0 = \alpha u_1 + (1 - \alpha) u_2 \in (\operatorname{int} Z_+) \times W_+$. Thereby,

$$m := \alpha m_1 + (1 - \alpha) m_2 \in \alpha J(x_1) + (1 - \alpha) J(x_2) + u_0.$$

Because of $\operatorname{int} Z_+ + Z_+ \subset \operatorname{int} Z_+$, so

$$m \in J(D) + U_+ + u_0 \subset J(D) + (\operatorname{int} Z_+) \times W_+ = M.$$

Therefore M is nearly convex. ∎

THEOREM 3.2. *Let* $\operatorname{int}(J(D) + U_+) \neq \emptyset$. *Suppose that* J *is nearly* U_+*-convexlike on* D. *Let* $H(D)$ *be a convex set with a nonempty topological interior. Then*

$$(Q_1) \Leftrightarrow (Q_2) \Leftrightarrow (Q_3).$$

Proof. We need only show $(Q_3) \Rightarrow (Q_1)$. Suppose that (i) of (Q_1) is not true. Then for any $x \in D$, we have $O \notin G(x) + \operatorname{int} Z_+$ or $O \notin H(x) + W_+$, i.e., $(O, O) \notin M = J(D) + (\operatorname{int} Z_+) \times W_+$. According to Lemma 3.1 and the assumption of $\operatorname{int}(J(D) + U_+) \neq \emptyset$, we have $\operatorname{int} M \neq \emptyset$. Furthermore, $(O, O) \notin \operatorname{int} M$. Since J is nearly U_+-convexlike on D, thus by Lemma 3.3, M is a nearly convex set. Hence, $\operatorname{int} M$ is convex. By applying the separation theorem of convex sets of linear topological spaces (see [19]), there is a hyperplane E properly separating $\{O\}$ and $\operatorname{int} M$, i.e., $\exists u^* = (z^*, w^*) \in Z^* \times W^* \setminus \{(O, O)\}$, $a \in R$, such that

$$\langle u, u^* \rangle \geq a \geq 0, \qquad \forall u \in \operatorname{int} M, \tag{3.1}$$

where the hyperplane $E = \{y \in U \mid \langle y, u^* \rangle = a\}$.

Next, we show that

$$\langle u, u^* \rangle > 0, \qquad \forall u \in \operatorname{int} M. \tag{3.2}$$

We have two cases. One case is $a > 0$. Then it follows by (3.1) and (3.2) holds. The other case is $a = 0$. We have

$$\langle u, u^* \rangle \geq 0, \qquad \forall u \in \operatorname{int} M. \tag{3.3}$$

Suppose that (3.2) does not hold. According to (3.3), there is $u_0 \in \operatorname{int} M$ such that $\langle u_0, u^* \rangle = 0$. Let $v \in \operatorname{int} M$. Then $\exists \varepsilon > 0$ such that $u_0 - \varepsilon v \in \operatorname{int} M$. Thus by (3.3), we have $\langle u_0 - \varepsilon v, u^* \rangle \geq 0$, i.e., $\langle u_0, u^* \rangle \geq \varepsilon \langle v, u^* \rangle$. So, $\langle v, u^* \rangle \leq 0$. On the other hand, again by (3.3) we get $\langle v, u^* \rangle \geq 0$. Therefore, $\langle v, u^* \rangle = 0$, $\forall v \in \operatorname{int} M$. This is absurd since the hyperplane E separates $\{O\}$ and $\operatorname{int} M$ properly. Hence the proof that (3.3) holds is complete.

By Lemma 2.3 we obtain

$$\langle u, u^* \rangle \geq 0, \qquad \forall u \in M. \tag{3.4}$$

In the following, we prove $u^* = (z^*, w^*) \in Z_+^* \times W_+^*$; indeed, assume $z^* \notin Z_+^*$. Then there is $z_1 \in Z_+$ such that $\langle z_1, z^* \rangle < 0$. By (3.4), for any $x \in D$, any $z' \in \operatorname{int} Z_+$, any $w' \in W_+$, we have $\beta = \langle p + z', z^* \rangle + \langle q + w', w^* \rangle \geq 0$, $\forall p \in G(x)$, $\forall q \in H(x)$. Since $\lambda z_1 \in Z_+$, then $\lambda z_1 + z' \in \operatorname{int} Z_+$. Also by (3.4), we have $\langle p + \lambda z_1 + z', z^* \rangle + \langle q + w', w^* \rangle \geq 0$, i.e.,

$$\lambda \langle z_1, z^* \rangle + \beta \geq 0, \qquad \forall \lambda > 0. \tag{3.5}$$

However, (3.5) does not hold when $\lambda > -\beta / \langle z_1, z^* \rangle \geq 0$. This contradiction illustrates that $z^* \in Z_+^*$. Similarly, we can also prove $w^* \in W_+^*$. Thus, $\exists u^* = (z^*, w^*) \in Z_+^* \times W_+^* \setminus \{(O, O)\}$, such that $\langle u, u^* \rangle \geq 0$, $\forall u \in M$, i.e.,

$$\langle G(x) + z, z^* \rangle + \langle H(x) + w, w^* \rangle \geq 0,$$
$$\forall x \in D, \forall z \in \operatorname{int} Z_+, \forall w \in W_+.$$

Take $w = O$ in the previous expression. Then

$$\langle G(x), z^* \rangle + \langle \lambda z, z^* \rangle + \langle H(x), w^* \rangle \geq 0,$$
$$\forall x \in D, \forall z \in \operatorname{int} Z_+, \forall \lambda > 0. \quad (3.6)$$

If $z^* = O$, then from (3.6) we have $\langle H(x), w^* \rangle \geq 0$, $\forall x \in D$, which contradicts (Q_3). Suppose $z^* \neq 0$. According to (Q_3), we have that $\exists x' \in D$, $p \in G(x')$, $q \in H(x')$ such that $\langle p, z^* \rangle + \langle q, w^* \rangle < 0$. Therefore there is a sufficiently small $\delta > 0$ such that

$$\langle p, z^* \rangle + \langle q, w^* \rangle + \delta < 0. \quad (3.7)$$

However, if we take $\lambda = \delta / \langle z, z^* \rangle > 0$ in (3.6), then (3.7) is in contradiction to (3.6). Therefore (i) of (Q_1) holds.

In the following we prove that (ii) of (Q_1) is true. Otherwise $\operatorname{int} H(D) \cap (-W_+) = \emptyset$. Thus again by the separation theorem of convex sets, there is $w^* \in W^* \backslash \{O\}$ such that $\langle H(D), w^* \rangle \geq \langle -W_+, w^* \rangle$, i.e., $\langle H(D), w^* \rangle + \langle W_+, w^* \rangle \geq 0$. It is easy to check that $w^* \in W_+^*$. Furthermore $\langle H(D), w^* \rangle \geq 0$, which contradicts (Q_3). ∎

THEOREM 3.3. *If* (Q_0) *holds, then there exists* $y_1 \in Y$, *such that* $(y_1, O, O) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$.

Proof. According to (Q_0), we have that there exists $x' \in D$ such that $O \in G(x') + \operatorname{int} Z_+ \subset \operatorname{int}(G(x') + Z_+)$, and $O \in \operatorname{int} H(x') + W_+ \subset \operatorname{int}(H(x') + W_+)$. Since $\emptyset \neq F(x')$, thus $\emptyset \neq F(x') + \operatorname{int} Y_+ \subset \operatorname{int}(F(x') + Y_+)$. Hence $\exists y_1 \in Y$ such that $(y_1, O, O) \in \operatorname{int}((F, G, H)(x') + Y_+ \times Z_+ \times W_+)$. ∎

Remark 3.1. In fact, Theorem 3.1, Theorem 3.2, and Theorem 3.3 are all still true when Y, Z, W are respectively assumed to be linear topological spaces.

4. OPTIMALITY CONDITIONS

In this section, we establish some K-T-type sufficient and necessary optimality conditions for set-valued optimization in terms of contingent epiderivatives which were introduced by Jahn and Rauh [1].

THEOREM 4.1 (Sufficiency). *Let* $D \subset X$ *be a convex set, let* $(F, G, H)(x)$ *be* $Y_+ \times Z_+ \times W_+$*-convex on* D, *and let* (x_0, y_0, z_0, w_0) *be a feasible point of* (P). *Suppose that* (Q_1) *holds. If the contingent epiderivative* $D(F, G, H)(x_0, y_0, z_0, w_0)$ *exists and there is* $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \backslash \{(O, O, O)\}$, *such that*

$$\left\langle D(F,G,H)(x_0,y_0,z_0,w_0)(x-x_0),(y^*,z^*,w^*)\right\rangle \geq 0, \qquad \forall x \in D,$$
$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0,$$

then x_0 *is a weakly efficient solution of* (P).

Proof. According to Lemma 3 of [1], for each $x \in D$ we have

$$(F,G,H)(x) - \{(y_0,z_0,w_0)\} \subset \{D(F,G,H)(x_0,y_0,z_0,w_0)(x-x_0)\} + Y_+ \times Z_+ \times W_+.$$

Thus for each $x \in D$, each $(y,z,w) \in (F,G,H)(x)$, there $c \in Y_+$, $d \in Z_+$, $e \in W_+$ such that

$$(y-y_0-c, z-z_0-d, w-w_0-e) = D(F,G,H)(x_0,y_0,z_0,w_0)(x-x_0).$$

Hence, by assumption, we obtain

$$\langle y-y_0-c, y^* \rangle + \langle z-z_0-d, z^* \rangle + \langle w-w_0-e, w^* \rangle \geq 0,$$
$$\text{or } \langle y-y_0, y^* \rangle + \langle z, z^* \rangle + \langle w, w^* \rangle$$
$$\geq \langle c, y^* \rangle + \langle z_0, z^* \rangle + \langle d, z^* \rangle + \langle w_0, w^* \rangle + \langle e, w^* \rangle$$
$$= \langle c, y^* \rangle + \langle d, z^* \rangle + \langle e, w^* \rangle \geq 0.$$

Therefore,

$$\langle y-y_0, y^* \rangle + \langle z, z^* \rangle + \langle w, w^* \rangle \geq 0,$$
$$\forall (y,z,w) \in (F,G,H)(x), \forall x \in D. \quad (4.1)$$

Next we show $y^* \neq O$. Suppose that $y^* = O$. Then $(z^*, w^*) \neq (O,O)$, and (4.1) can be written as

$$\left\langle G(x), z^* \right\rangle + \left\langle H(x), w^* \right\rangle \geq 0, \qquad \forall x \in D. \quad (4.2)$$

Thus we have two cases. One case is $z^* \neq 0$. By Q_1, $\exists x' \in D$, $\exists z' \in G(x')$, $w' \in H(x')$, such that $-z' \in \operatorname{int} Z_+$, $-w' \in W_+$. Hence, it follows by Lemma 3.2 that $\langle z', z^* \rangle + \langle w', w^* \rangle < 0$, which contradicts (4.2). The other case is $z^* = 0$. Then $w^* \neq O$. Again by Q_1, there is $w^{\#} \in W_+$ such that $-w^{\#} \in \operatorname{int} H(D)$. Then for any $w \in W$, $\exists \lambda > 0$ such that $-w^{\#} \pm \lambda w \in H(D)$. So, $\langle -w^{\#} \pm \lambda w, w^* \rangle \geq 0$, i.e., $\pm \lambda \langle w, w^* \rangle \geq \langle w^{\#}, w^* \rangle \geq 0$. Therefore, $\langle w, w^* \rangle = 0$. It follows that $w^* = 0$. This is a contradiction. Therefore we have $y^* \neq 0$.

Finally, we prove x_0 is a weakly efficient solution of (P1). Otherwise, $\exists x^{\#} \in K$ such that $(y_0 - F(x^{\#})) \cap \operatorname{int} Y_+ \neq \emptyset$. Thus, $\exists t \in F(x^{\#})$ such that $y_0 - t \in \operatorname{int} Y_+$. By Lemma 2.2, we have

$$\langle t - y_0, y^* \rangle < 0. \tag{4.3}$$

Since $x^{\#} \in K$, there are $p \in G(x^{\#})$, $q \in H(x^{\#})$ such that $-p \in Z_+$, $-q \in W_+$. Taking (4.3) into account, we get $\langle t - y_0, y^* \rangle + \langle p, z^* \rangle + \langle q, w^* \rangle < 0$, which contradicts (4.1). ∎

In the remainder of this paper, we assume that $\operatorname{int} W_+ \neq \emptyset$.

THEOREM 4.2 (Necessity). *Let $D \subset Y$ be a convex set, let (F, G, H) be $Y_+ \times Z_+ \times W_+$-convex on D, and let (x_0, y_0, z_0, w_0) be a weak minimizer of* (P). *Suppose that* Q_0 *is fulfilled. If the contingent epiderivative $D(F, G, H)(x_0, y_0, z_0, w_0)$ exists, and $T_X(\operatorname{epi}(F, G, H), (x_0, y_0, z_0, w_0)) = X$, then there is $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$, such that*

$$\left\langle D(F,G,H)(x_0, y_0, z_0, w_0)(x - x_0), (y^*, z^*, w^*) \right\rangle \geq 0, \qquad \forall x \in D,$$
$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0.$$

Proof. Set

$$P(x) = D(F,G,H)(x_0, y_0, z_0, w_0)(x - x_0), \qquad x \in D.$$

We show first that $P(x)$ is a $Y_+ \times Z_+ \times W_+$-convex single-valued map on D. Let $x_i \in D$, $i = 1, 2$, $\lambda \in (0,1)$. Thus by positive homogeneity, we have

$$\lambda P(x_1) = D(F,G,H)(x_0, y_0, z_0, w_0)(\lambda x_1 - x_0),$$
$$(1 - \lambda) P(x_2) = D(F,G,H)(x_0, y_0, z_0, W_0)((1 - \lambda) x_2 - x_0).$$

By subadditivity, we obtain

$$\lambda P(x_1) + (1 - \lambda) P(x_2) \in P\left(\lambda x_1 + (1 - \lambda) x_2\right) + Y_+ \times Z_+ \times W_+.$$

This illustrates that $P(x)$ is $Y_+ \times Z_+ \times W_+$-convex on D.

Set $R(x) = P(x) + (O, z_0, w_0)$, $\forall x \in D$. Obviously, $R(x)$ is also $Y_+ \times Z_+ \times W_+$-convex on D. Next we prove that

$$-R(x) \notin \operatorname{int} Y_+ \times \operatorname{int} Z_+ \times \operatorname{int} W_+, \qquad \forall x \in D.$$

Otherwise, there is $x' \in D$ such that

$$-R(x') := -(y', z', w') - (O, z_0, w_0) \in \operatorname{int} Y_+ \times \operatorname{int} Z_+ \times \operatorname{int} W.$$

By definition of the contingent epiderivative,

$$(x' - x_0, y', z', w') \in \operatorname{epi}\big(D(F,G,H)(x_0, y_0, z_0, w_0)\big)$$
$$= T\big(\operatorname{epi}(F,G,H), (x_0, y_0, z_0, w_0)\big).$$

According to the definition of the contingent cone, the following sequences exist: $x_n \in D$, $c_n \in Y_+$, $d_n \in Z_+$, $e_n \in W_+$, $y_n \in F(x_n) + c_n$, $z_n \in G(x_n) + d_n$, $w_n \in H(x_n) + e_n$, satisfying $x_n \to x_0$, $y_n \to y_0$, $z_n \to z_0$; $\lambda_n > 0$, such that

$$\lambda_n(x_n - x_0, y_n - y_0, z_n - z_0, w_n - w_0) \to (x' - x_0, y', z', w').$$

Thus, $\lambda_n(y_0 - y_n) \to -y'$, $\lambda_n(z_0 - z_n) \to -z'$, $\lambda_n(w_0 - w_n) \to -w'$. Because $-y' \in \operatorname{int} Y_+$, so $\exists M_1 \in N$ such that $\lambda_n(y_0 - y_n) \in \operatorname{int} Y_+$, $\forall n \geq M_1$. Since $\operatorname{int} Y_+$ is a cone, and since $\lambda_n > 0$, hence

$$y_0 - y_n \in \operatorname{int} Y_+, \qquad \forall n \geq M_1. \tag{4.4}$$

Because $-z' - z_0 \in \operatorname{int} Z_+$, so $\exists M_2 \in N$ such that $\lambda_n(z_0 - z_n) - z_0 \in \operatorname{int} Z_+$, $\forall n \geq M_2$. That is $\lambda_n((1 - 1/\lambda_n)z_0 - z_n) \in \operatorname{int} Z_+$, $\forall n \geq M_2$, or $(1 - 1/\lambda_n)z_0 - z_n \in \operatorname{int} Z_+$, $\forall n \geq M_2$. In a similar way, $\exists M_3 \in N$ such that $(1 - 1/\lambda_n)w_0 - w_n \in \operatorname{int} W_+$, $\forall n \geq M_3$.

Take $M = \max(M_1, M_2, M_3)$. Whenever $n \geq M$, we have $\lambda_n > k > 1$. Otherwise $\lambda_n(y_0 - y_n) \to O \in \operatorname{int} Y_+$ because of $y_n \to y_0$. This is absurd. Since $(1 - 1/\lambda_n)z_0 \in -Z_+$, and since $\operatorname{int} Z_+ + Z_+ \subset \operatorname{int} Z_+$, therefore $-z_n \in \operatorname{int} Z_+ \subset Z_+$. So $-z_n \in -(G(x_n) + Z_+) \cap Z_+$, $\forall n \geq M$. Since Z_+ is a convex cone, we get

$$-G(x_n) \cap Z_+ \neq \emptyset, \qquad \forall n \geq M.$$

Similarly, we have $-H(x_n) \cap W_+ \neq \emptyset$, $\forall n \geq M$. This means that x_n belongs to a feasible set of (P). It follows by (4.4) that

$$\big(y_0 - F(x_n) - Y_+\big) \cap \operatorname{int} Y_+ \neq \emptyset, \qquad \forall n \geq M,$$

since $y_n \in F(x_n) + Y_+$. So, $(y_0 - F(x_n)) \cap \operatorname{int} Y_+ \neq \emptyset$, which does not coincide with the supposition that (x_0, y_0, z_0, w_0) is a weak minimizer of (P). Therefore,

$$-R(x) \notin \operatorname{int} Y_+ \times \operatorname{int} Z_+ \times \operatorname{int} W_+, \qquad \forall x \in D.$$

By applying Lemma 2.2 of [13] or Theorem 3.1 of [12], we have that $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$, such that

$$\big\langle R(x), (y^*, z^*, w^*)\big\rangle \geq 0, \qquad \forall x \in D.$$

Hence,

$$\langle P(x), (y^*, z^*, w^*) \rangle + \langle z_0, z^* \rangle + \langle w_0, w^* \rangle \geq 0, \qquad \forall x \in D,$$

where $P(x) = D(F, G, H)(x_0, y_0, z_0, w_0)(x - x_0)$. Since $(O, O, O) = P(x_0)$, hence $\langle z_0, z^* \rangle + \langle w_0, w^* \rangle \geq 0$. On the other hand, due to $z_0 \in -Z_+$, $w_0 \in -W_+$, consequently, $\langle z_0, z^* \rangle + \langle w_0, w^* \rangle \leq 0$. Therefore, $\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0$. Thus $\langle P(x), (y^*, z^*, w^*) \rangle \geq 0$, $\forall x \in D$, i.e.,

$$\langle D(F,G,H)(x_0, y_0, z_0, w_0)(x - x_0), (y^*, z^*, w^*) \rangle \geq 0, \qquad \forall x \in D.$$

Next we show $y^* \neq O$. Since Q_0 is fulfilled, then from Theorem 3.3, there exists $y_1 \in Y$ such that $(y_1, O, O) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$. Again by Lemma 3 of [1], we have

$$(F,G,H)(x) - \{(y_0, z_0, w_0)\} \subset \{D(F,G,H)(x_0, y_0, z_0, w_0)(x - x_0)\} + Y_+ \times Z_+ \times W_+, \qquad \forall x \in D.$$

Thus

$$\langle y - y_0, y^* \rangle + \langle z, z^* \rangle + \langle w, w^* \rangle \geq 0, \quad \forall (y, z, w) \in (F, G, H)(x), \forall x \in D. \tag{4.5}$$

Suppose that $y^* = O$. Then $(z^*, w^*) \neq 0$, and (4.5) can be written as

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \qquad \forall x \in D. \tag{4.6}$$

Thus,

$$\langle F(x) + Y_+, y^* \rangle + \langle G(x) + Z_+, z^* \rangle + \langle H(x) + W_+, w^* \rangle \geq 0, \quad \forall x \in D. \tag{4.7}$$

Since $(y_1, O, O) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$, then for any $x \in D$, $v_1 \in G(x)$, $v_2 \in H(x)$, $k_1 \in Z_+$, $k_2 \in W_+$, there is $\varepsilon > 0$ such that $(y_1, O, O) \pm \varepsilon(O, v_1 + k_1, v_2 + k_2) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$. By (4.7), we have $\langle v_1 + k_1, z^* \rangle + \langle v_2 + k_2, w^* \rangle = 0$. Observing (4.6), we get

$$\langle k_1, z^* \rangle + \langle k_2, w^* \rangle = 0, \qquad \forall k_1 \in Z_+, \forall k_2 \in W_+. \tag{4.8}$$

Then we also have two cases. One case is $z^* \neq 0$. Then (4.8) contradicts Lemma 3.2. The other case is $z^* = 0$. Then $w^* \neq 0$. For any $v \in W$, there is $\varepsilon_1 > 0$ such that $(y_0, O, O) \pm \varepsilon_1(O, O, v) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$. By (4.7), we obtain $\langle v, w^* \rangle = 0$, $\forall v \in W$. Thus $w^* = 0$. This is a contradiction. Therefore we have $y^* \neq 0$. ∎

COROLLARY 4.1. *Let $D \subset Y$ be a convex set, let $(F, G, H)(x)$ be $Y_+ \times Z_+ \times W_+$-convex on D, and let (x_0, y_0, z_0, w_0) be a weak minimizer of* (P). *Suppose that there exists $y_1 \in Y$ such that $(y_1, O, O) \in \operatorname{int}((F, G, H)(D) + Y_+ \times Z_+ \times W_+)$. If the contingent epiderivative $D(F, G, H)(x_0, y_0, z_0, w_0)$ exists, and $T_X(\operatorname{epi}(F, G, H), (x_0, y_0, z_0, w_0)) = X$, then there is $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$, such that*

$$\langle D(F, G, H)(x_0, y_0, z_0, w_0)(x - x_0), (y^*, z^*, w^*)\rangle \geq 0, \qquad \forall x \in D;$$

$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0.$$

COROLLARY 4.2. *Let $D \subset Y$ be a convex set, let $(F, G, H)(x)$ be $Y_+ \times Z_+ \times W_+$-convex on D, and let (x_0, y_0, z_0, w_0) be a weak minimizer of* (P). *Suppose that* Q_1 *is fulfilled. If the contingent epiderivative $D(F, G, H)(x_0, y_0, z_0, w_0)$ exists, and $T_X(\operatorname{epi}(F, G, H), (x_0, y_0, z_0, w_0)) = X$, then there is $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$, such that*

$$\langle D(F, G, H)(x_0, y_0, z_0, w_0)(x - x_0), (y^*, z^*, w^*)\rangle \geq 0, \qquad \forall x \in D;$$

$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0.$$

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