

## A Doppler Robust Max-Min Approach to Radar Code Design

Antonio De Maio, Yongwei Huang, and Marco Piezzo

**Abstract**—This correspondence considers the problem of robust waveform design in the presence of colored Gaussian disturbance under a similarity and an energy constraint. We resort to a max-min approach, where the worst case detection performance (over the possible Doppler shifts) is optimized with respect to the radar waveform under the previously mentioned constraints. The resulting optimization problem is a non-convex Quadratically Constrained Quadratic Program (QCQP) with an infinite number of constraints, which is NP-hard in general and typically difficult to solve. Hence, we propose an algorithm with a polynomial computational complexity to generate a *good* sub-optimal solution for the aforementioned QCQP. The analysis, conducted in comparison with some known radar waveforms, shows that the sub-optimal solutions by the algorithm lead to high-quality radar signals.

**Index Terms**—Non-convex quadratic optimization, non-negative trigonometric polynomials, radar waveform design, semidefinite programming relaxation, waveform diversity.

### I. INTRODUCTION

The advent of adaptive radar transmitters, which permit the use of advanced and flexible pulse shaping techniques, and the significant achievements in high speed signal processing hardware are paving the way to the development of very innovative and computational demanding techniques for radar waveform design [1], [2]. The idea is to adapt and diversify dynamically the transmitted signal to the operating environment in order to achieve a performance gain over classic radar waveforms [3]–[8].

In [9], focusing on the class of linearly coded pulse trains (both in amplitude and in phase), the authors introduce a code selection algorithm which maximizes the detection performance but, at the same time, is capable of controlling both the region of achievable values for the Doppler estimation accuracy and the degree of similarity with a pre-fixed radar code. However, since in several practical situations, the radar amplifiers might work in saturation conditions and hence an amplitude modulation might be difficult to perform, in [10], the authors also consider the synthesis of constant modulus phase coding schemes for radar coherent pulse trains. Finally, in [11], the problem of constrained code optimization for radar space-time adaptive processing (STAP) in the presence of colored Gaussian disturbance, under two accuracy constraints (on the temporal and the spatial Doppler frequency) and a similarity constraint, is addressed.

Many among the previously mentioned algorithms optimize the radar signal in correspondence of a given target Doppler frequency. Hence, they can be easily applied to situations where it is required a confirmation of an initial detection in a certain Doppler bin, namely

Manuscript received December 09, 2009; accepted May 05, 2010. Date of publication May 17, 2010; date of current version August 11, 2010. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Xiang-Gen Xia.

A. De Maio and M. Piezzo are with Università degli Studi di Napoli “Federico II”, Dipartimento di Ingegneria Biomedica Elettronica e delle Telecomunicazioni, I-80125 Napoli, Italy (e-mail: ademai@unina.it; marpiez@libero.it).

Y. Huang is with Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong (e-mail: eeyw@ust.hk).

Color versions of one or more of the figures in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TSP.2010.2050317

when some knowledge about the Doppler frequency is available. In other situations, the Doppler parameter is usually unknown and a practical application of the techniques can be obtained either tuning the design Doppler to a challenging condition, dictated by the clutter power spectral density (PSD) shape, or optimizing the waveform to an average scenario, namely considering as objective function the average SNR over the possible target Doppler shifts. This correspondence moves another step towards the synthesis of radar waveforms when no prior knowledge about the actual Doppler is available. Specifically, resorting to the max-min criterion, we formulate the waveform design problem as the constrained maximization of the worst case (over the set of possible Doppler frequencies) detection performance. The constraints considered here are an energy constraint, imposed by the finite transmission resources, and a similarity constraint, important to equip the waveform with desirable properties such as small modulus variations, good range resolution, low peak sidelobe levels, and more in general with a good ambiguity function. The resulting problem is a non-convex Quadratically Constrained Quadratic Program (QCQP) with infinitely many quadratic constraints. This class of QCQPs, is known to be NP-hard in general, and as a consequence, finding a global optimal solution is often very difficult [12]. Hence, the present work aims to the construction of a *good* sub-optimal solution for the quoted problem with the goodness in the sense that the produced solution leads to an high-quality radar code for our robust radar waveform design problem, as our simulations illustrate in Section IV.

The correspondence is organized as follows. In Section II, we present the system model and formulate the waveform design problem according to the max-min criterion; in Section III, we introduce the new algorithm for the considered problem; in Section IV, we analyze the performance of the proposed technique and provide numerical results assessing the quality of the produced sub-optimal solution. Finally, conclusions are given in Section V.

### II. SYSTEM MODEL AND WAVEFORM DESIGN PROBLEM

We consider the same signal model as in [9]. Precisely, the  $N$ -dimensional column vector containing the samples of the received signal (after down-conversion and matched filtering)  $\mathbf{v} = [v(t_0), v(t_1), \dots, v(t_{N-1})]^T$  can be written as<sup>1</sup>

$$\mathbf{v} = \alpha \mathbf{c} \odot \mathbf{p} + \mathbf{w} \quad (1)$$

where  $\alpha$  is the complex echo amplitude (accounting for the transmit amplitude, phase, target reflectivity, and channels propagation effects),  $\mathbf{c} = [a(0), a(1), \dots, a(N-1)]^T$  is the  $N$ -dimensional column vector containing the transmitted code elements,  $\mathbf{p} = (1/\sqrt{N})[1, e^{j2\pi\nu_d}, \dots, e^{j2\pi(N-1)\nu_d}]^T$  is the temporal steering vector,  $\nu_d$  denotes the normalized Doppler frequency,  $\odot$  is the Hadamard element-wise product [13], and  $\mathbf{w} = [w(t_0), w(t_1), \dots, w(t_{N-1})]^T$  is the vector containing the disturbance samples.

We are looking for a radar waveform which optimizes the worst case detection performance, under an energy constraint and a similarity constraint with a given radar code exhibiting a good ambiguity function. In this section, we formulate mathematically this problem showing how the worst case detection probability can be maximized and the constraints can be enforced, under the assumption that  $\mathbf{w}$  is a zero-mean complex circular Gaussian vector with known positive definite covariance matrix  $E[\mathbf{w}\mathbf{w}^\dagger] = \mathbf{R}$ , ( $E[\cdot]$  denotes statistical expectation and  $(\cdot)^\dagger$  conjugate transpose). It is known [9] that the detection probability ( $P_d$ ) of the generalized likelihood ratio test (GLRT), for a given

<sup>1</sup> $(\cdot)^T$  is the transpose operator.

value of the false alarm probability  $P_{fa}$ , depends on the radar code, the disturbance covariance matrix, and the temporal steering vector only through the SNR, defined as

$$\text{SNR} = |\alpha|^2 (\mathbf{c} \odot \mathbf{p})^\dagger \mathbf{M} (\mathbf{c} \odot \mathbf{p}), \quad (2)$$

( $|\cdot|$  indicates the modulus of a complex number) which is a function of the actual Doppler frequency due to the dependence of  $\mathbf{p}$  over  $\nu_d$ . Moreover,  $P_d$  is an increasing function of SNR and, as a consequence, the maximization of  $P_d$  can be obtained maximizing the quadratic form

$$(\mathbf{c} \odot \mathbf{p})^\dagger \mathbf{M} (\mathbf{c} \odot \mathbf{p}) = \mathbf{c}^\dagger (\mathbf{M} \odot (\mathbf{p}\mathbf{p}^\dagger)^*) \mathbf{c} \quad (3)$$

over the radar code ( $(\cdot)^*$  denotes complex conjugate). We just highlight that  $\mathbf{M} \odot (\mathbf{p}\mathbf{p}^\dagger)^*$  is the Hadamard product of two positive semidefinite matrices, and hence it is itself positive semidefinite [14, p. 1352, A.77].

Performing the maximization of (3), possibly under some constraints [9] (for instance accuracy, similarity, and energy constraints), leads to a code vector which depends on the specific value of the Doppler frequency present in the definition of  $\mathbf{p}$ . In order to get a transmit radar waveform independent of the Doppler frequency, we propose here a max-min approach attempting at maximizing the worst case (over the possible target Doppler frequencies) SNR. In other words, we consider as objective function to maximize over the radar code

$$\min_{\nu_d \in [0,1]} \mathbf{c}^\dagger (\mathbf{M} \odot (\mathbf{p}\mathbf{p}^\dagger)^*) \mathbf{c}.$$

Adding the similarity constraint with a code  $\mathbf{c}_0$  [8], important to confer desirable properties to the radar waveform, as well as an energy constraint (accounting for the limited transmission power), we come up with the following optimization problem:

$$\max_{\mathbf{c} \in \Omega} \min_{\nu_d \in [0,1]} \mathbf{c}^\dagger (\mathbf{M} \odot (\mathbf{p}\mathbf{p}^\dagger)^*) \mathbf{c} \quad (4)$$

where the set  $\Omega$  is defined as  $\Omega = \{\mathbf{c} \mid \|\mathbf{c}\| = 1, \|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon\}$  with  $\|\mathbf{c}_0\| = 1$  ( $\|\cdot\|$  is the Euclidean norm), and the parameter  $\epsilon \geq 0$  ruling the size of the similarity region. Indeed, the smaller  $\epsilon$  is, the higher the degree of similarity between the ambiguity functions of the designed radar code and  $\mathbf{c}_0$  is.

Before presenting the new algorithm, we would point out the differences between this optimization problem and those formulated and solved in [9] and [10]. To this end, we observe that the objective function in [9] and [10] depends on a specific design Doppler value, while in the present problem the worst case SNR (over the Doppler frequency) is optimized (4). In [9], we account for a Doppler dependent constraint on the estimation accuracy of  $f_d$ , while in the present case, only a similarity and an energy constraint are considered. In [10], we account for a phase-only constraint on the devised code, while in this paper a general amplitude-phase coding is considered. In other words, (4) optimizes a robust objective function with respect to [9] and [10], but the former forces one less quadratic constraint than the problem in [9], and the constraints of the problem specified in [10] look very different from those in (4). From the optimization theory point of view, the three formulations lead to different optimization problems:

- 1) that in [9] is a homogeneous QCQP with three constraints, a global optimal solution for which can be found in polynomial time (namely for this problem the SDP relaxation is tight or, equivalently, the problem shares a hidden convexity);
- 2) that in [10] is an NP-hard QCQP optimization problem due to the phase-only and the possibly finite alphabet constraint, whose optimal solution is approximated using the relaxation and randomization approach typical of the max-cut-like problems;
- 3) that in the current correspondence is a QCQP with infinitely many constraints, for which we establish a deterministic approxima-

tion procedure, with polynomial time computational complexity, to output a solution leading to high-quality radar waveforms.

### III. APPROXIMATE SOLUTION TO THE MAX-MIN OPTIMIZATION PROBLEM

The max-min problem (4) can be recast as

$$\begin{cases} \max_{\mathbf{c}, t} & t \\ \text{s.t.} & t \leq \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{c}\mathbf{c}^\dagger)^*) \mathbf{p}, \quad \forall \nu_d \in [0, 1], \\ & \|\mathbf{c} - \mathbf{c}_0\|^2 \leq \epsilon, \\ & \|\mathbf{c}\| = 1. \end{cases} \quad (5)$$

Moreover, elaborating on the similarity constraint, problem (5) can be equivalently rewritten as

$$\begin{cases} \max_{\mathbf{c}, t} & t \\ \text{s.t.} & t \leq \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{c}\mathbf{c}^\dagger)^*) \mathbf{p}, \quad \forall \nu_d \in [0, 1], \\ & \text{Re}(\mathbf{c}^\dagger \mathbf{c}_0) \geq 1 - \epsilon/2, \\ & \|\mathbf{c}\|^2 = 1 \end{cases} \quad (6)$$

with  $\text{Re}(\cdot)$  the real part of the argument. Observing that a rotation of  $\mathbf{c}$  does not change the first constraint, we claim that problem (6) is equivalent to

$$\begin{cases} \max_{\mathbf{c}, t} & t \\ \text{s.t.} & t \leq \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{c}\mathbf{c}^\dagger)^*) \mathbf{p}, \quad \forall \nu_d \in [0, 1], \\ & \mathbf{c}^\dagger \mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{c} \geq \delta_\epsilon, \\ & \|\mathbf{c}\|^2 = 1 \end{cases} \quad (7)$$

where  $\delta_\epsilon = (1 - \epsilon/2)^2$ , in the sense that if  $(\mathbf{c}^*, t^*)$  is an optimal solution of problem (7), then  $(\mathbf{c}^* e^{j \arg \mathbf{c}_0^\dagger \mathbf{c}_0}, t^*)$  is an optimal solution of (6)<sup>2</sup>. Therefore, let us focus on problem (7) from now on.

It is seen that problem (7) is a QCQP with infinitely many constraints. As already highlighted, this class of problems is known to be NP-hard in general (see [12]) and hence difficult to solve. In other words, the convex relaxation of the class of QCQP problem may or may not be tight, in particular, its SDP relaxation may have only optimal solutions of rank higher than one, or may have optimal solutions of rank higher than one as well as equal to one. Further, to retrieve a rank-one optimal solution of the SDP relaxation problem from an optimal solution of general rank is usually a non-trivial task. In the following, we will present an approximation scheme to produce a feasible solution for the problem (7), based on the techniques of SDP relaxation, SDP representation of trigonometric polynomials, and a specific rank-one matrix decomposition. It turns out by our numerical simulations that the algorithm provides high-quality radar codes for our robust waveform design problem. Additionally, if the SDP relaxation is tight (namely, the SDP has always a rank-one optimal solution) then the devised code is also optimal for the original non-convex problem.

The SDP relaxation of (7) is<sup>3</sup>

$$\begin{cases} \max_{\mathbf{C}, t} & t \\ \text{s.t.} & t \leq \mathbf{p}^\dagger (\mathbf{M} \odot \mathbf{C}^*) \mathbf{p}, \quad \forall \nu_d \in [0, 1], \\ & \text{tr}(\mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{C}) \geq \delta_\epsilon, \\ & \text{tr}(\mathbf{C}) = 1, \\ & \mathbf{C} \succeq \mathbf{0} \end{cases} \quad (8)$$

where  $\text{tr}(\cdot)$  is the trace of a square matrix and  $\mathbf{0}$  is a matrix (of suitable size) with all zero entries. Clearly, the constraint function  $\mathbf{p}^\dagger (\mathbf{M} \odot \mathbf{C}^*) \mathbf{p} - t$  is a trigonometric polynomial [16] of degree  $N - 1$ , that is,

$$\mathbf{p}^\dagger (\mathbf{M} \odot \mathbf{C}^*) \mathbf{p} - t = x_0 - t + 2 \text{Re} \left( \sum_{k=1}^{N-1} x_k e^{-jk\omega} \right)$$

<sup>2</sup> $\arg(\cdot)$  is the argument operator.

<sup>3</sup> $\mathbf{C} \succeq \mathbf{0}$  means that  $\mathbf{C}$  is Hermitian positive semidefinite [15].

where  $\omega = 2\pi\nu_d$  and

$$x_k = \frac{1}{N} \sum_{i=1}^{N-k} (\mathbf{M} \odot \mathbf{C}^*)(i+k, i), \quad k = 0, 1, \dots, N-1, \quad (9)$$

with the notation  $(\mathbf{M} \odot \mathbf{C}^*)(i+k, i)$  being the  $(i+k, i)$ th entry of  $\mathbf{M} \odot \mathbf{C}^*$ .

It is known that the nonnegativity constraint of a trigonometric polynomial has an equivalent SDP representation. Specifically, the following result derived in [17, Theorem 3.1] is quoted here as a lemma.

**Lemma 3.1:** The trigonometric polynomial  $f(\omega) = x_0 + 2\text{Re}(\sum_{k=1}^{N-1} x_k e^{-jk\omega})$  is non-negative over  $[0, 2\pi]$ , if and only if there exists an  $N \times N$  Hermitian matrix  $\mathbf{X}$  such that

$$\mathbf{x} = \mathbf{W}^\dagger \text{diag}(\mathbf{W}\mathbf{X}\mathbf{W}^\dagger), \quad \mathbf{X} \succeq \mathbf{0} \quad (10)$$

where  $\mathbf{x} = [x_0, \dots, x_{N-1}]^T$ ,  $\mathbf{W} = [\mathbf{w}_0, \dots, \mathbf{w}_{N-1}] \in \mathbb{C}^{M \times N}$ ,  $\mathbf{w}_k = [1, e^{-jk\theta}, \dots, e^{-j(M-1)k\theta}]^T$ ,  $k = 0, \dots, N-1$ ,  $\theta = 2\pi/M$ ,  $M \geq 2N-1$ , and  $\text{diag}(\cdot)$  denotes the vector containing the diagonal elements of the square matrix argument.

It follows by Lemma 3.1 that SDP (8) is equivalent to the SDP

$$\begin{cases} \max_{\mathbf{X}, \mathbf{C}, t} & t \\ \text{s.t.} & \mathbf{W}^\dagger \text{diag}(\mathbf{W}\mathbf{X}\mathbf{W}^\dagger) + t\mathbf{e}_1 = \mathbf{x}, \\ & \text{tr}(\mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{C}) \geq \delta_\epsilon, \\ & \text{tr}(\mathbf{C}) = 1, \\ & \mathbf{C} \succeq \mathbf{0}, \\ & \mathbf{X} \succeq \mathbf{0} \end{cases} \quad (11)$$

where  $\mathbf{x}$  is defined by (9),  $\mathbf{e}_1$  is the  $N$ -dimensional vector with the first component being one and the others zero,  $\mathbf{W}$  is the same as the one defined in Lemma 3.1 by taking  $M = 2N-1$ . In order to proceed further it is necessary to show the following:

**Lemma 3.2:** It holds that SDP problem (11) is solvable.<sup>4</sup>

*Proof:* The proof is based on the Conic Duality Theorem [18, Theorem 1.7.1] and is omitted due to the lack of space.

Let  $(\mathbf{X}^*, \mathbf{C}^*, t^*)$  be an optimal solution of (11). It is easily seen that  $(\mathbf{C}^*, t^*)$  is an optimal solution of SDP (8) with

$$t^* = \min_{\nu_d \in [0, 1]} \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{C}^*)^*) \mathbf{p}. \quad (12)$$

Problem (12) is one dimensional optimization problem with sufficiently smooth objective function, therefore we can apply Newton method to solve it. Letting

$$\nu_d^* = \arg \min_{\nu_d \in [0, 1]} \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{C}^*)^*) \mathbf{p}, \quad (13)$$

namely a value of  $\nu_d \in [0, 1]$  minimizing the argument and

$$\mathbf{p}^* = \frac{1}{\sqrt{N}} [1, e^{j2\pi\nu_d^*}, \dots, e^{j(N-1)2\pi\nu_d^*}]^T, \quad (14)$$

we have

$$t^* = \mathbf{p}^{\dagger} (\mathbf{M} \odot (\mathbf{C}^*)^*) \mathbf{p}^* = \text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^*) \mathbf{C}^*].$$

Now if  $\mathbf{C}^*$  is rank-one, namely  $\mathbf{C}^* = \mathbf{c}_1 \mathbf{c}_1^\dagger$ , then  $\mathbf{c}^* = \mathbf{c}_1 e^{j \arg \mathbf{c}_1^\dagger \mathbf{c}_0}$  and  $\nu_d^*$  are optimal for the original max-min problem, i.e., the SDP relaxation is tight. Otherwise, we can provide an approximate solution to (4). To this end, we wish to find a rank-one matrix  $\mathbf{c}\mathbf{c}^\dagger$  such that

$$\text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^*) \mathbf{c}\mathbf{c}^\dagger] = \text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^*) \mathbf{C}^*] = t^*, \quad (15)$$

<sup>4</sup>By saying ‘‘solvable,’’ we mean that the problem is feasible, bounded, and the optimal value is attained (see [18, p. 13]).

$$\text{tr}(\mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{c}\mathbf{c}^\dagger) = \text{tr}(\mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{C}^*) = s, \quad (16)$$

$$\text{tr}(\mathbf{c}\mathbf{c}^\dagger) = \text{tr}(\mathbf{C}^*) = 1, \quad (17)$$

as long as  $\mathbf{C}^*$  is of rank higher than one. If we can find a rank-one solution  $\mathbf{c}\mathbf{c}^\dagger$  satisfying (15)–(17), then the following one-dimensional search yields a feasible solution of problem (7):

$$\nu_d = \arg \min_{\nu_d \in [0, 1]} \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{c}\mathbf{c}^\dagger)^*) \mathbf{p}, \quad (18)$$

with the optimal value

$$t = \min_{\nu_d \in [0, 1]} \mathbf{p}^\dagger (\mathbf{M} \odot (\mathbf{c}\mathbf{c}^\dagger)^*) \mathbf{p}. \quad (19)$$

In other words,  $(\mathbf{c}, t)$  is a sub-optimal solution of problem (7). To find a rank-one solution of (15)–(17), we resort to the following rank-one decomposition theorem [19].

**Lemma 3.3:** Suppose that  $\mathbf{X}$  is an  $N \times N$  complex Hermitian positive semidefinite matrix of rank  $R$ , and  $\mathbf{A}_1, \mathbf{A}_2$  are two  $N \times N$  given Hermitian matrices. Then, there is a rank-one decomposition of  $\mathbf{X}$  (synthetically denoted as  $\mathcal{D}(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2)$ ),  $\mathbf{X} = \sum_{r=1}^R \mathbf{x}_r \mathbf{x}_r^\dagger$ , such that

$$\begin{aligned} \mathbf{x}_r^\dagger \mathbf{A}_1 \mathbf{x}_r &= \frac{\text{tr}(\mathbf{X} \mathbf{A}_1)}{R} \quad \text{and} \\ \mathbf{x}_r^\dagger \mathbf{A}_2 \mathbf{x}_r &= \frac{\text{tr}(\mathbf{X} \mathbf{A}_2)}{R}, \quad r = 1, \dots, R. \end{aligned}$$

In our context, it is necessary to perform  $\mathcal{D}(\mathbf{C}^*, \mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^* - t^* \mathbf{I}, \mathbf{c}_0 \mathbf{c}_0^\dagger - s \mathbf{I})$  obtaining  $\mathbf{C}^* = \sum_{i=1}^R \mathbf{c}_i \mathbf{c}_i^\dagger$ , where  $R = \text{Rank}(\mathbf{C}^*)$ . Then, it is easily verified that each  $\mathbf{c}_i \mathbf{c}_i^\dagger / \|\mathbf{c}_i\|^2$  for  $i = 1, \dots, R$ , fulfills (15)–(17). In fact,

$$\frac{1}{R} \text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^* - t^* \mathbf{I}) \mathbf{c}_i \mathbf{c}_i^\dagger] = \text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^* - t^* \mathbf{I}) \mathbf{C}^*] = 0, \quad (20)$$

$$\frac{1}{R} \text{tr}[(\mathbf{c}_0 \mathbf{c}_0^\dagger - s \mathbf{I}) \mathbf{c}_i \mathbf{c}_i^\dagger] = \text{tr}[(\mathbf{c}_0 \mathbf{c}_0^\dagger - s \mathbf{I}) \mathbf{C}^*] = 0, \quad (21)$$

which imply

$$\text{tr}[(\mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^*) \mathbf{c}_i \mathbf{c}_i^\dagger] = t^* \|\mathbf{c}_i\|^2, \quad (22)$$

$$\text{tr}[(\mathbf{c}_0 \mathbf{c}_0^\dagger) \mathbf{c}_i \mathbf{c}_i^\dagger] = s \|\mathbf{c}_i\|^2. \quad (23)$$

As a consequence,  $\mathbf{c}_i \mathbf{c}_i^\dagger / \|\mathbf{c}_i\|^2$ , for  $i = 1, \dots, R$ , complies with (15)–(17). Performing the one-dimensional optimization problem (19) gives the sub-optimal solutions  $(\mathbf{c}_i / \|\mathbf{c}_i\|, t_i)$ , where  $t_i$  is the optimal value of problem (19) corresponding to  $\mathbf{c}_i / \|\mathbf{c}_i\|$ . Take the maximal value of  $\{t_1, \dots, t_R\}$ , say  $t_1$ , and output  $(\mathbf{c}_1 / \|\mathbf{c}_1\|, t_1)$  as the sub-optimal solution (namely the best among the couples  $(\mathbf{c}_i / \|\mathbf{c}_i\|, t_i)$ ).

Summarizing, a sub-optimal solution for problem (4) can be generated as follows.

---

**Algorithm 1: Approximation Procedure for the Max-Min Problem (4)**

---

**Input:**  $\mathbf{c}_0, \epsilon, \mathbf{M}, N$ ;

**Output:** a sub-optimal solution  $(\mathbf{c}^*, \nu_d^*)$  of problem (4);

- 1: solve SDP (11) finding  $(\mathbf{X}^*, \mathbf{C}^*, t^*)$ ;
  - 2: solve problem (12) obtaining  $\nu_d^*$ ; compute  $\mathbf{p}^*$  like (14);
  - 3: let  $\text{tr}(\mathbf{c}_0 \mathbf{c}_0^\dagger \mathbf{C}^*) = s$ , and perform  $\mathcal{D}(\mathbf{C}^*, \mathbf{M} \odot (\mathbf{p}^* \mathbf{p}^{*\dagger})^* - t^* \mathbf{I}, \mathbf{c}_0 \mathbf{c}_0^\dagger - s \mathbf{I})$  getting  $\mathbf{C}^* = \sum_{i=1}^R \mathbf{c}_i \mathbf{c}_i^\dagger$ ;
  - 4: let  $\mathbf{c}_i = \mathbf{c}_i / \|\mathbf{c}_i\|$ ,  $i = 1, \dots, R$ , and solve problem (19) with parameter  $\mathbf{c}_i$ , obtaining the optimal values  $\{t_1, \dots, t_R\}$  and the optimum  $\{\nu_{d,1}, \dots, \nu_{d,R}\}$ ;
  - 5: choose  $\mathbf{c}_i$  such that  $t_i = \max\{t_1, \dots, t_R\}$ , say  $\mathbf{c}_i = \mathbf{c}_1$ , and let  $\mathbf{c}^* = \mathbf{c}_1 e^{j \arg \mathbf{c}_1^\dagger \mathbf{c}_0}$  and  $\nu_d^* = \nu_{d,1}$ .
-

As to the computational complexity of the above algorithm, it is dictated by the solution of the SDP problem (11),<sup>5</sup> which has a worst-case complexity of  $O(N^{4.5} \log(1/\eta))$  (see [18]), since the specific rank-one decomposition involved requires  $O(N^3)$  operations and the cost of the one dimensional optimization problem<sup>6</sup> is very low compared to the cost of the computations in the other steps.

Before concluding, we highlight that a possible extension of the encoding algorithm aimed at optimizing the minimum SNR (over  $\nu_d$ ) in a subinterval of  $[0, 1]$  (or even in the union of more than one of such subintervals) can be easily conceived exploiting [17, Theorem 3.2] in place of [17, Theorem 3.1] to express the nonnegativity of the trigonometric polynomial in the considered subinterval.

#### IV. PERFORMANCE ANALYSIS

This section is devoted to the performance analysis of the proposed scheme for the robust waveform design. To this end, we assume that the  $(l, k)$ th entry disturbance covariance matrix is given by  $\mathbf{R}(l, k) = \rho_1^{|l-k|} \exp[j2\pi\gamma(l-k)] + 10\rho^{|l-k|} + 10^{-2}\mathbf{I}(l, k)$ , which is a structure accounting for the simultaneous presence of sea clutter, land clutter, and thermal noise. Moreover, we fix  $P_{fa}$  of the GLRT receiver to  $10^{-6}$ ,  $\rho_1 = 0.8$ ,  $\rho = 0.9$ , and  $\gamma = 0.2$ . The analysis is conducted in terms of  $P_d$ , robustness with respect to Doppler shifts, and ambiguity function of the coded pulse train which results exploiting the proposed algorithm, i.e.,

$$\begin{aligned} \chi(\lambda, f) &= \int_{-\infty}^{\infty} u(\beta)u^*(\beta - \lambda)e^{j2\pi f\beta} d\beta \\ &= \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \bar{a}(l)\bar{a}^*(m)\chi_p[\lambda - (l-m)T_r, f] \end{aligned}$$

where  $[\bar{a}(0), \dots, \bar{a}(N-1)]$  is an optimized code. The convex optimization MATLAB toolbox SELF-DUAL-MINIMIZATION (SeDuMi) [20] is exploited for solving the SDP relaxation. The decomposition  $\mathcal{D}(\cdot, \cdot, \cdot)$  of the SeDuMi solution is performed using the technique described in [19]. Finally, the MATLAB toolbox of [21] is used to plot the ambiguity functions of the coded pulse trains. In the following, we consider as similarity code the generalized Barker sequence [21, pp. 109-113] of length  $N = 10$   $\mathbf{c}_0 = [0.3162, 0.3162, 0.1724 + 0.2651j, -0.1905 + 0.2524j, -0.2322 + 0.2147j, 0.3084 + 0.0697j, 0.3141 + 0.0367j, -0.2250 - 0.2222j, 0.29851 + 0.1044j, -0.1881 - 0.2542j]^T$ . In Fig. 1, we plot  $P_d$  of the optimized code (according to the max-min criterion) versus  $|\alpha|^2$  for several values of  $\delta_\epsilon$ , together with  $P_d$  of the similarity code for  $\nu_d = \nu_d^*$ . The curves show that increasing  $\delta_\epsilon$  worse and worse  $P_d$  values are obtained; this behavior can be explained observing that the smaller  $\delta_\epsilon$ , the larger  $\epsilon$ , the larger the size of the similarity region. However, this detection loss is compensated for an improvement of the coded pulse train ambiguity function. This is shown in Fig. 2(a)–(d), where such function is plotted assuming rectangular pulses,  $T_r = 5T_p$ . The plots highlight that the closer  $\delta_\epsilon$  to 1 the higher the degree of similarity between the ambiguity functions of the devised and the pre-fixed code. This is due to the fact that increasing  $\delta_\epsilon$  is tantamount to reducing the size of the similarity region. In other words, we force the devised code to be similar and similar to the pre-fixed one and, as a consequence, we get similar and similar ambiguity functions.

The last analysis of this section concerns the robustness of  $P_d$  with respect to Doppler shifts. Specifically, we plot, in Fig. 3,  $P_d$  versus  $\nu_d$  for the max-min code and the similarity code  $\mathbf{c}_0$ , assuming  $|\alpha|^2 = 23$

<sup>5</sup>An SDP problem can be efficiently solved in polynomial time through *interior point methods*, and the number of iterations necessary to achieve convergence usually ranges between 10 and 100 (see [15]).

<sup>6</sup>In the later numerical simulation, we use the Matlab command `fminbnd`.

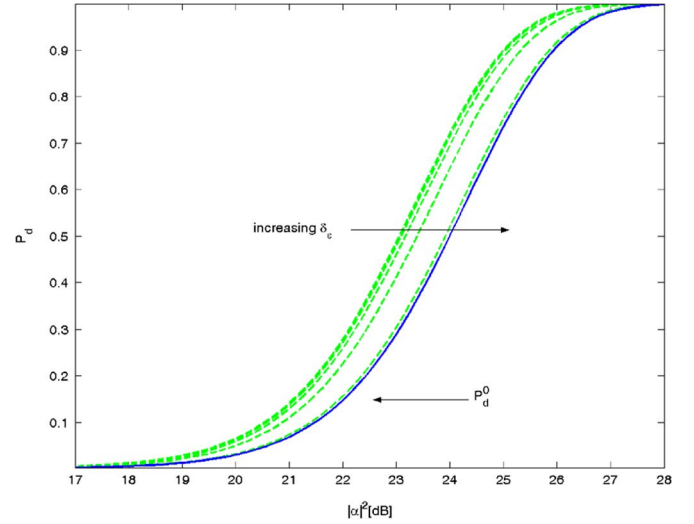


Fig. 1.  $P_d$  versus  $|\alpha|^2$  for non-fluctuating target,  $P_{fa} = 10^{-6}$ ,  $N = 10$ ,  $\nu_d = \nu_d^*$ , and  $\delta_\epsilon = \{0.1, 0.4, 0.7, 0.9, 0.9801, 0.9999\}$ . Generalized Barker code (solid curve). Max-min code (dashed curves).

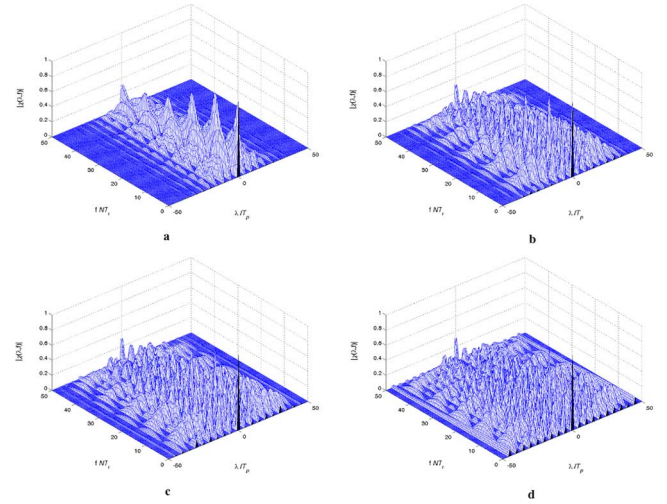


Fig. 2. (a) Ambiguity function modulus of the max-min code with  $N = 10$ ,  $T_r = 5T_p$ ,  $\delta_\epsilon = 0.4$ . (b) Ambiguity function modulus of the max-min code with  $N = 10$ ,  $T_r = 5T_p$ ,  $\delta_\epsilon = 0.7$ . (c) Ambiguity function modulus of the max-min code with  $N = 10$ ,  $T_r = 5T_p$ ,  $\delta_\epsilon = 0.9$ . (d) Ambiguity function modulus of the max-min code with  $N = 10$ ,  $T_r = 5T_p$ ,  $\delta_\epsilon = 0.9999$ .

dB. Inspection of the curves highlights that, for values of  $\delta_\epsilon \leq 0.9$ ,  $P_d$  of the optimized code exhibits a quite flat behavior with respect to Doppler frequencies. On the contrary,  $P_d$  of the similarity code is very sensitive to the Doppler shift and exhibits significant variations. Moreover, for a wide range of Doppler shifts the max-min code outperforms the similarity sequence. Actually, the smaller  $\delta_\epsilon$ , the wider the Doppler interval where the max-min code performs better than the similarity code  $\mathbf{c}_0$ .

Now, we provide a numerical analysis aimed at assessing the quality of the solution produced by the new algorithm. Specifically, we evaluate the normalized gap  $\Delta_g$  between the optimal value of the SDP problem and  $t_1$ , i.e.,  $\Delta_g = (t^* - t_1)/t^*$ . Observing the second row of Table I, we can see that, for the considered values of the parameters, the devised algorithm provides high-quality solutions. We highlight that, for all the simulated  $\delta_\epsilon \geq 0.7$  or  $0.15 \leq \delta_\epsilon < 0.4$ , it even outputs the optimal solution to the max-min problem (i.e., the SDP relaxation problem has always a rank-one optimal solution).

TABLE I  
 $\Delta_g$  FOR  $N = 10$ , SEVERAL VALUES OF  $\delta_\epsilon$ , AND GENERALIZED BARKER CODE AS SIMILARITY SEQUENCE

$\delta_\epsilon$	0.4	0.45	0.47	0.5	0.53	0.55	0.57	0.6	0.63	0.65	0.67
$\Delta_g$	0.22%	1.39%	1.89%	2.69%	3.56%	4.08%	4.54%	5.16%	5.67%	5.15%	2.75%

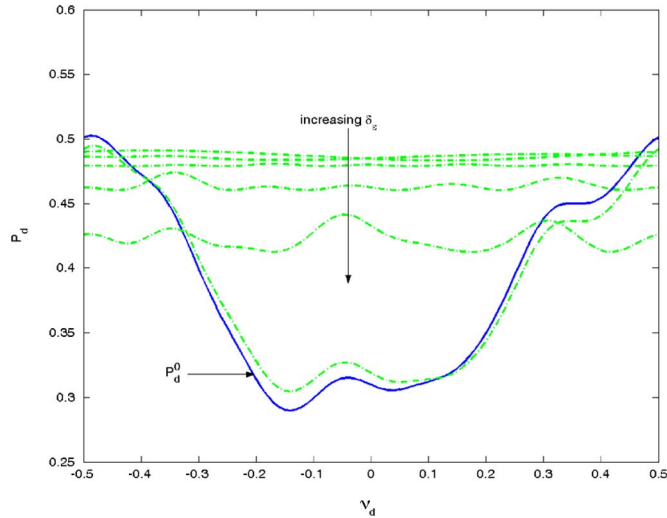


Fig. 3.  $P_d$  versus  $\nu_d$  for  $|\alpha|^2 = 23$  dB, non-fluctuating target,  $N = 10$ , and  $\delta_\epsilon = \{0.1, 0.4, 0.7, 0.9, 0.9801, 0.9999\}$ . Generalized Barker code (solid curves), Max-min code (dash curves).

## V. CONCLUSION

We herein have proposed and analyzed a max-min algorithm for radar waveform design, in the presence of colored Gaussian disturbance, forcing energy and similarity constraints. The waveform synthesis has been formulated as a non-convex quadratic optimization problem with infinitely many quadratic constraints. Through a clever technique, exploiting SDP relaxation techniques and some results from the theory of non-negative trigonometric polynomials, we devised a procedure capable of providing a high-quality waveform from an optimal solution of the SDP relaxation. The technique is based on a suitable rank-one decomposition and its implementation requires a polynomial computational complexity. At the analysis stage, we have evaluated the performance of the new algorithm in terms of detection performance, ambiguity function and robustness of detection probability with respect to Doppler shifts. The effect of the similarity parameter has been studied. Precisely, if there are sufficient degrees of freedom for the optimization problem, namely the similarity parameter is not close to 0, then the max-min algorithm is capable of ensuring a very robust detection performance with respect to target Doppler shifts. Moreover, this robust behavior can be traded off with ambiguity function peculiarities.

## REFERENCES

- [1] A. Farina, "Waveform Diversity: Past, Present, and Future," presented at the 3rd Int. Waveform Diversity Design Conf., Pisa, Italy, Jun. 2007, plenary talk.
- [2] M. Wicks, "A brief history of waveform diversity," presented at the 2009 Radar Conf., Pasadena, CA, May 2009.
- [3] A. Nehorai, F. Gini, M. S. Greco, A. Papandreou-Suppappola, and M. Rangaswamy, Eds., *IEEE J. Sel. Topics Signal Process. Special Issue on Adaptive Waveform Design for Agile Sensing and Communications*, vol. 1, no. 1, pp. 2–213, Jun. 2007.

- [4] A. Farina and F. A. Studer, "Detection with high resolution radar: Great promise, big challenge," *Microw. J.*, pp. 263–273, May 1991.
- [5] S. U. Pillai, H. S. Oh, D. C. Youla, and J. R. Guerci, "Optimum transmit-receiver design in the presence of signal-dependent interference and channel noise," *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 577–584, Mar. 2000.
- [6] S. Kay, "Optimal signal design for detection of point targets in stationary Gaussian clutter/reverberation," *IEEE J. Sel. Topics Signal Process.*, vol. 1, no. 1, pp. 31–41, Jun. 2007.
- [7] J. S. Bergin, P. M. Techau, J. E. Don Carlos, and J. R. Guerci, "Radar waveform optimization for colored noise mitigation," in *Proc. IEEE Int. Radar Conf.*, Alexandria, VA, May 9–12, 2005, pp. 149–154.
- [8] J. Li, J. R. Guerci, and L. Xu, "Signal waveform's optimal-under-restriction design for active sensing," *IEEE Signal Process. Lett.*, vol. 13, no. 9, pp. 565–568, Sep. 2006.
- [9] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, "Code design to optimize radar detection performance under accuracy and similarity constraints," *IEEE Trans. Signal Process.*, vol. 56, no. 11, pp. 5618–5629, Nov. 2008.
- [10] A. De Maio, S. De Nicola, Y. Huang, Z. Q. Luo, and S. Zhang, "Design of phase codes for radar performance optimization with a similarity constraint," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 610–621, Feb. 2009.
- [11] A. De Maio, S. De Nicola, Y. Huang, D. Palomar, S. Zhang, and A. Farina, "Code design for radar STAP via optimization theory," *IEEE Trans. Signal Process.*, vol. 58, no. 2, pp. 679–694, Feb. 2010.
- [12] Z. Q. Luo and T. H. Chang, "SDP relaxation of homogeneous quadratic optimization: Approximation bounds and applications," in *Convex Optimization in Signal Processing and Communications*, D. P. Palomar and Y. Eldar, Eds. Cambridge, U.K.: Cambridge Univ. Press, 2010, ch. 4.
- [13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.
- [14] H. L. Van Trees, *Optimum Array Processing. Part IV of Detection, Estimation and Modulation Theory*. New York: Wiley, 2002.
- [15] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2003.
- [16] B. Dumitrescu, *Positive Trigonometric Polynomials and Signal Processing Applications*. New York: Springer, 2007.
- [17] T. Roh and L. Vandenberghe, "Discrete transforms, semidefinite programming, and sum-of-squares representations of nonnegative polynomials," *SIAM J. Optim.*, vol. 16, no. 4, pp. 939–964, 2006.
- [18] A. Nemirovski, Lectures on modern convex optimization, Class notes, Georgia Inst. of Technol., Atlanta, GA, Fall 2005.
- [19] Y. Huang and S. Zhang, "Complex matrix decomposition and quadratic programming," *Math. Oper. Res.*, vol. 32, no. 3, pp. 758–768, Aug. 2007.
- [20] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Methods Softw.*, vol. 11–12, pp. 625–653, Aug. 1999.
- [21] N. Levanon and E. Mozeson, *Radar Signals*. New York: Wiley, 2004.