Introduction to Convex Optimization

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Outline of Lecture

- Optimization problems
- Examples
- Solving optimization problems
- More examples
- Course goals
- References

(Acknowledgement to Stephen Boyd for material for this lecture.)

Optimization Problem

- General optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

  where

  $x = (x_1, \ldots, x_n)$ is the optimization variable
  $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ is the objective function
  $f_i : \mathbf{R}^n \longrightarrow \mathbf{R}, \quad i = 1, \ldots, m$ are inequality constraint functions
  $h_i : \mathbf{R}^n \longrightarrow \mathbf{R}, \quad i = 1, \ldots, p$ are equality constraint functions.

- **Goal**: find an optimal solution $x^\star$ that minimizes f_0 while satisfying all the constraints.

Examples

Convex optimization is currently used in many different areas:

- circuit design (start-up named Barcelona in Silicon Valley)
- signal processing (e.g., filter design)
- communication systems (e.g., transceiver design, beamforming design, ML detection, power control in wireless)
- financial engineering (e.g., portfolio design, index tracking)
- image proc. (e.g., deblurring, compressive sensing, blind separation)
- machine learning
- biomedical applications (e.g., analysis of DNA)

Examples: Elements in the Formulation

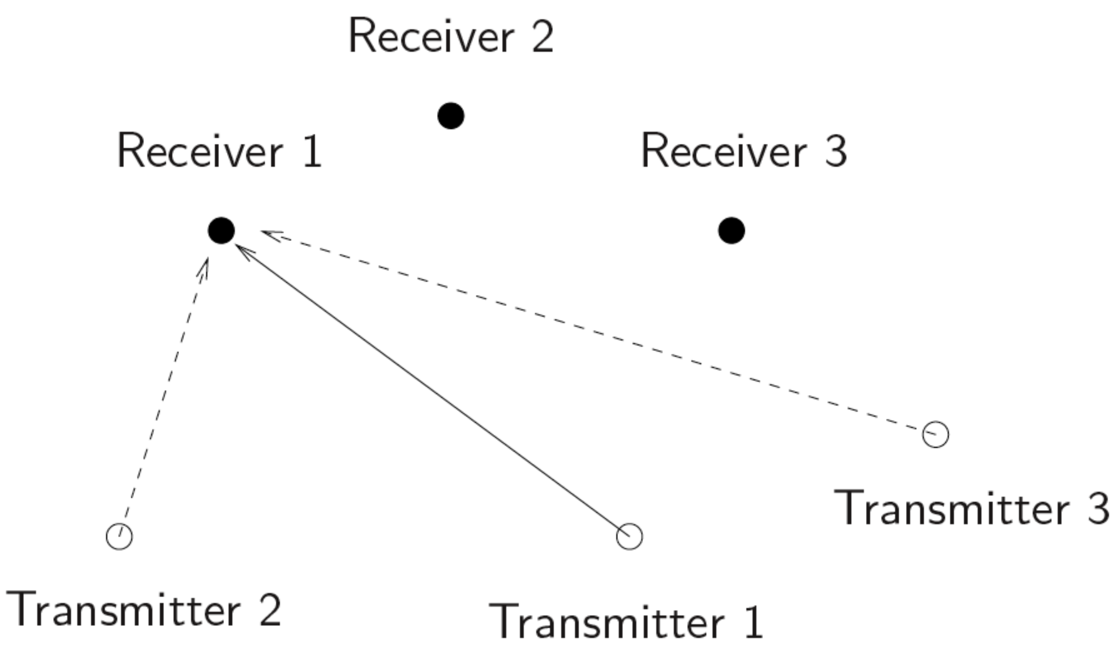
- An optimization problem has three basic elements: 1) variables, 2) constraints, and 3) objective.

- Example: device sizing in electronic circuits:
  - variables: device widths and lengths
  - constraints: manufacturing limits, timing requirements, max area
  - objective: power consumption

- Example: portfolio optimization:
  - variables: amounts invested in different assets
  - constraints: budget, max investments per asset, min return
  - objective: overall risk or return variance.

Example: Power Control in Wireless Networks

- Consider a wireless network with n logical transmitter/receiver pairs:



- Goal: design the power allocation so that each receiver receives minimum interference from the other links.

- The signal-to-inerference-plus-noise-ratio (SINR) at the ith receiver is

$$\mathsf{sinr}_i = \frac{p_i G_{ii}}{\sum_{j \neq i} p_j G_{ij} + \sigma_i^2}$$

where

p_i is the power used by the ith transmitter
G_{ij} is the path gain from transmitter j to receiver i
σ_i^2 is the noise power at the ith receiver.

- **Problem**: maximize the weakest SINR subject to power constraints $0 \leq p_i \leq p_i^{\max}$:

$$\begin{array}{ll} \underset{\mathbf{p}}{\text{maximize}} & \min\limits_{i=1,\ldots,n} \frac{p_i G_{ii}}{\sum_{j \neq i} p_j G_{ij} + \sigma_i^2} \\ \text{subject to} & 0 \leq p_i \leq p_i^{\max} \qquad i = 1, \ldots, n. \end{array}$$

Solving Optimization Problems

- General optimization problems are very difficult to solve (either long computation time or not finding the best solution).
- Exceptions: least-squares problems, linear programming problems, and convex optimization problems.
- **Least-squares (LS)**:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2$$

  - solving LS problems: closed-form solution $x^\star = \left(A^T A\right)^{-1} A^T b$ for which there are reliable and efficient algorithms; mature technology
  - using LS: easy to recognize

- **Linear Programming (LP)**:
$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - solving LP problems: no closed-form solution, but reliable and efficient algorithms and software; mature technology
  - using LP: not as easy to recognize as LS problems, a few standard tricks to convert problems into LPs

- **Convex optimization**:
$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - solving convex problems: no closed-form solution, but still reliable and efficient algorithms and software; almost a technology
  - using convex optimization: often difficult to recognize, many tricks for transforming problems into convex form.

Nonconvex Optimization

Nonconvex optimization problems are generally very difficult to solve, although there are some rare exceptions.

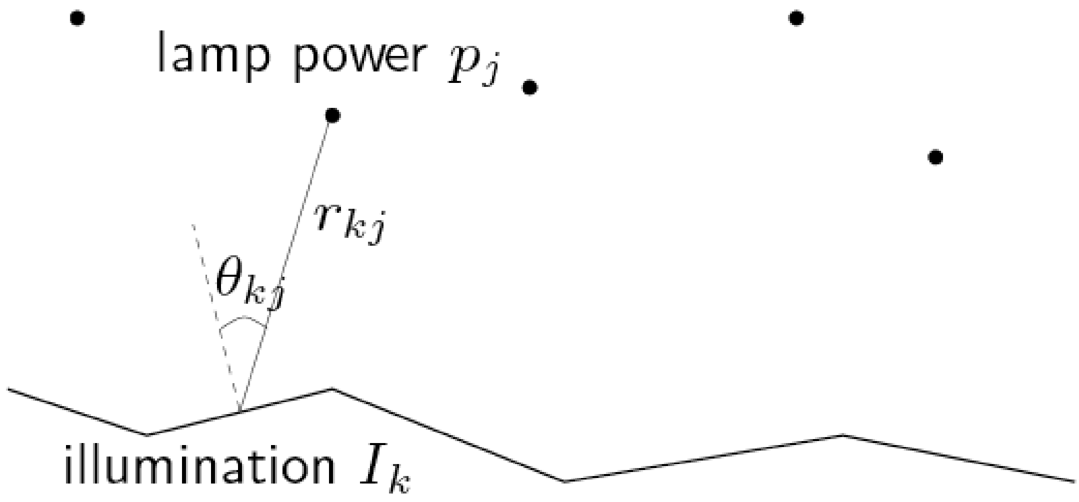
In general, they require either a long computation time or the compromise of not always finding the optimal solution:

- local optimization: fast algorithms, but no guarantee of global optimality, only local solution around the initial point

- global optimization: worst-case complexity grows exponentially with problem size, but finds global solution.

Example: Lamp Illumination Problem

- Consider m lamps illuminating n small flat patches:



- Goal: achieve a desired illumination I_{des} on all patches with bounded lamp powers.

- The intensity I_k at patch k depends linearly on the lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{kj} p_j$$

  where the coefficients a_{kj} are given by $a_{kj} = \cos\theta_{kj}/r_{kj}^2$.

- Problem formulation: since the illumination is perceived logarithmically by the eye, a good formulation of the problem is

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k |\log I_k - \log I_{\mathsf{des}}| \\ \text{subject to} & 0 \le p_j \le p_{\max}, \quad j = 1,\ldots,m \\ & I_k = \sum_{j=1}^{m} a_{kj} p_j, \quad k = 1,\ldots,n. \end{array}$$

- How to solve the problem? The answer is: it depends on how much you know about optimization.

Solving the problem:

1. If you don't know anything, then you just take a heuristic guess like using a uniform power $p_j = p$, perhaps trying different values of p.

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2. If you know about least-squares, then approximate the problem as

$$\underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} \quad \sum_{k=1}^{n} \left(I_k - I_{\text{des}}\right)^2$$

and then clip p_j if $p_j > p_{\max}$ or $p_j < 0$.

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   and then clip p_j if $p_j > p_{\max}$ or $p_j < 0$.

3. If you know about linear programming, then approximate the problem as

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k \left|I_k - I_{\text{des}}\right| \\ \text{subject to} & 0 \le p_j \le p_{\max}, \quad j = 1, \ldots, m. \end{array}$$

4. If you know about convex optimization, after staring at the problem long enough, you may realize that you can actually reformulate the original problem in convex form and then find the global solution:

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k h\left(I_k/I_{\mathsf{des}}\right) \\ \text{subject to} & 0 \leq p_j \leq p_{\max}, \quad j = 1,\ldots,m. \end{array}$$

where $h\left(u\right) = \max\left\{u, 1/u\right\}$.

- **Additional constraints**: does adding the constraints below complicate the problem?

  (a) no more than half of total power is in any 10 lamps
  (b) no more than half of the lamps are on $(p_j > 0)$.

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  (a) no more than half of total power is in any 10 lamps
  (b) no more than half of the lamps are on ($p_j > 0$).

- Answer: adding (a) does not complicate the problem, whereas adding (b) makes the problem extremely difficult.

- Moral: untrained intuition doesn't always work; one needs to obtain the proper background and develop the right intuition to discern between difficult and easy problems.

History Snapshop of Convex Optimization

- **Theory** (convex analysis): ca1900-1970 (e.g. Rockafellar)
- **Algorithms**:
  - 1947: simplex algorithm for linear programming (Dantzig)
  - 1960s: early interior-point methods (Fiacco & McCormick, Dikin)
  - 1970s: ellipsoid method and other subgradient methods
  - 1980s: polynomial-time interior-point methods for linear programming (Karmakar 1984)
  - late 1980s-now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)
- **Applications**:
  - before 1990s: mostly in operations research; few in engineering
  - since 1990: many new applications in engineering and new problem classes (SDP, SOCP, robust optim.)

Course Goals and Topics

- **Goal**: to introduce convex optimization theory and to illustrate its use with many recent applications with emphasis on

  i) the art of unveiling the hidden convexity of problems

  ii) a proper characterization of the solution either analytically or algorithmically.

- The course follows a case-study approach by considering applications such as filter/beamforming design, circuit design, robust designs under uncertainty, portfolio optimization in finance, transceiver design for MIMO channels, image processing, blind separation, design of multiple access channels, multiuser detection, duality in information theory, network optimization, distributed algorithms, wireless network power control, machine learning, etc.

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