Convex Problems

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ELEC5470 - Convex Optimization
Fall 2018-19, HKUST, Hong Kong

Outline of Lecture

- Optimization problems
- Convex optimization (def., optimality conditions, reformulations)
- Quasi-convex optimization
- Classes of convex problems: LP, QP, SOCP, SDP.
- Multicriterion optimization (Pareto optimality)

(Acknowledgement to Stephen Boyd for material for this lecture.)

Optimization Problem in Standard Form

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

$x \in \mathbf{R}^n$ is the optimization variable

$f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ is the objective function

$f_i : \mathbf{R}^n \longrightarrow \mathbf{R}$, $i = 1, \ldots, m$ are inequality constraint functions

$h_i : \mathbf{R}^n \longrightarrow \mathbf{R}$, $i = 1, \ldots, p$ are equality constraint functions.

- **Feasibility**:
  - a point $x \in \mathrm{dom}\, f_0$ is feasible if it satisfies all the constraints and infeasible otherwise
  - a problem is feasible if it has at least one feasible point and infeasible otherwise.

- **Optimal value**:

$$p^\star = \inf \left\{ f_0\left(x\right) \mid f_i\left(x\right) \leq 0,\ i = 1, \ldots, m,\ h_i\left(x\right) = 0,\ i = 1, \ldots, p \right\}$$

  $p^\star = \infty$ if problem infeasible (no x satisfies the constraints)
  $p^\star = -\infty$ if problem unbounded below.

- **Optimal solution**: $x^\star$ such that $f\left(x^\star\right) = p^\star$ (and $x^\star$ feasible).

Global and Local Optimality

- A feasible x is **optimal** if $f_0(x) = p^\star$; X_{opt} is the set of optimal points.

- A feasible x is **locally optimal** if is optimal within a ball, i.e., there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \underset{z}{\text{minimize}} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0,\ i = 1, \ldots, m,\ \ h_i(z) = 0,\ i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

Examples:

- $f_0(x) = 1/x$, dom $f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point

- $f_0(x) = x^3 - 3x$: $p^\star = -\infty$, local optimum at $x = 1$.

Implicit Constraints

- The standard form optimization problem has an explicit constraint:

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom } f_i \ \cap \ \bigcap_{i=1}^{p} \text{dom } f_i$$

- $\mathcal{D}$ is the domain of the problem

- The constraints $f_i(x) \leq 0, \ h_i(x) = 0$ are the explicit constraints

- A problem is unconstrained if it has no explicit constraints

- Example:

$$\underset{x}{\text{minimize}} \quad \log\left(b - a^T x\right)$$

  is an unconstrained problem with implicit constraint $b > a^T x$.

Feasibility Problem

- Sometimes, we don't really want to minimize any objective, just to find a feasible point:

$$\begin{array}{lll} \underset{x}{\text{find}} & x & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

- This feasibility problem can be considered as a special case of a general problem:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & 0 & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

where $p^\star = 0$ if constraints are feasible and $p^\star = \infty$ otherwise.

Convex Optimization Problem

- Convex optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

  where $f_0, f_1, \ldots, f_m$ are convex and equality constraints are affine.

- **Local and global optima**: any locally optimal point of a convex problem is globally optimal.

- Most problems are not convex when formulated.

- Reformulating a problem in convex form is an art, there is no systematic way.

Example

- The following problem is nonconvex (why not?):

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1/\left(1+x_2^2\right) \leq 0 \\ & \left(x_1+x_2\right)^2 = 0 \end{array}$$

- The objective is convex.
- The equality constraint function is not affine; however, we can rewrite it as $x_1 = -x_2$ which is then a linear equality constraint.
- The inequality constraint function is not convex; however, we can rewrite it as $x_1 \leq 0$ which again is linear.
- We can rewrite it as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 = -x_2 \end{array}$$

Global and Local Optimality

Any locally optimal point of a convex problem is globally optimal.

Proof: Suppose x is locally optimal (around a ball of radius R) and y is optimal with $f_0(y) < f_0(x)$. We will show this cannot be.

Just take the segment from x to y: $z = \theta y + (1-\theta)x$. Obviously the objective function is strictly decreasing along the segment since $f_0(y) < f_0(x)$:

$$\theta f_0(y) + (1-\theta) f_0(x) < f_0(x) \qquad \theta \in (0,1].$$

Using now the convexity of the function, we can write

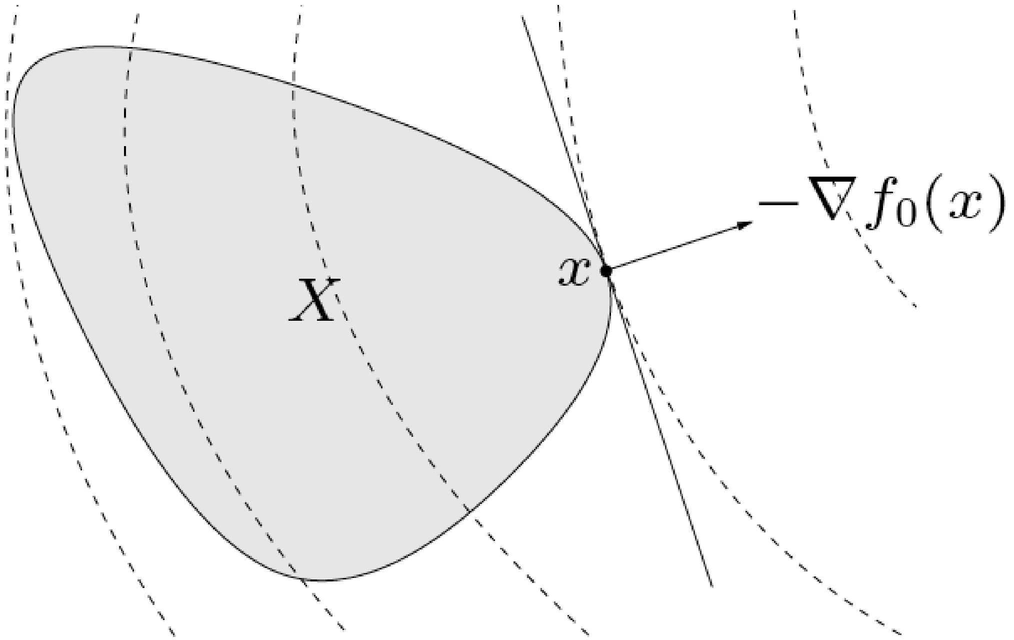
$$f_0(\theta y + (1-\theta)x) < f_0(x) \qquad \theta \in (0,1].$$

Finally, just choose θ sufficiently small such that the point z is in the ball of local optimality of x, arriving at a contradiction.

Optimality Criterion for Differentiable f_0

Minimum Principle: A feasible point x is optimal if and only if

$$\nabla f_0(x)^T (y - x) \geq 0 \qquad \text{for all feasible } y$$



- **unconstrained problem**: x is optimal iff

$$x \in \mathsf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

- **equality constrained problem**: $\underset{x}{\mathsf{min}}\ f_0(x) \quad \mathsf{s.t.}\ Ax = b$

x is optimal iff

$$x \in \mathsf{dom}\, f_0, \qquad Ax = b,\ \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**: $\underset{x}{\mathsf{min}}\ f_0(x) \quad \mathsf{s.t.}\ x \geq 0$

x is optimal iff

$$x \in \mathsf{dom}\, f_0, \qquad x \geq 0, \begin{cases} \nabla_i f_0(x) \geq 0 & x_i = 0 \\ \nabla_i f_0(x) = 0 & x_i > 0 \end{cases}$$

Equivalent Reformulations

- **Eliminating/introducing equality constraints**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{z}{\text{minimize}} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0 \qquad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that $Ax = b \Longleftrightarrow x = Fz + x_0$ for some z.

- **Introducing slack variables for linear inequalities**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & a_i^T x \le b_i \qquad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{x,s}{\text{minimize}} & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i \qquad i = 1, \ldots, m \\ & s_i \ge 0 \end{array}$$

- **Epigraph form**: a standard form convex problem is equivalent to

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0 \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- **Minimizing over some variables**:

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f_0(x, y) \\ \text{subject to} & f_i(x) \leq 0 \qquad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \tilde{f}_0(x) \\ \text{subject to} & f_i(x) \leq 0 \qquad i = 1, \ldots, m \end{array}$$

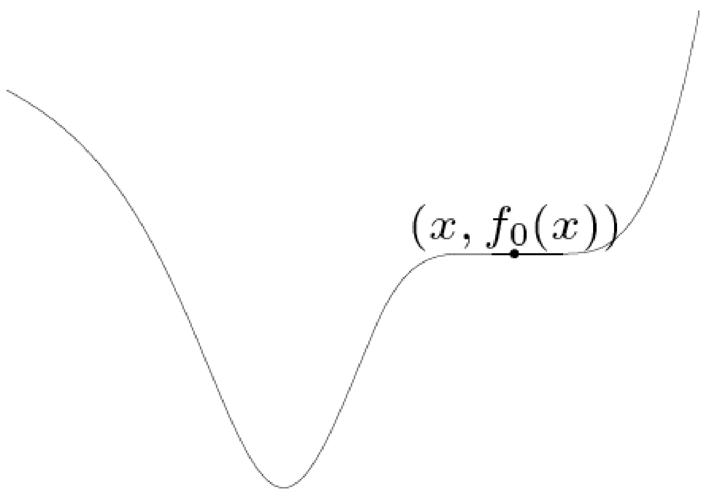
where $\tilde{f}_0(x) = \inf_y f_0(x, y)$.

Quasiconvex Optimization

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

where $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ is quasiconvex and $f_1, \ldots, f_m$ are convex.

- Observe that it can have locally optimal points that are not (globally) optimal:



- **Convex representation** of sublevel sets of a quasiconvex function f_0: there exists a family of convex functions $\phi_t(x)$ for fixed t such that

$$f_0(x) \leq t \quad \Longleftrightarrow \quad \phi_t(x) \leq 0.$$

- **Example**:

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0$, $q(x) > 0$ on $\mathrm{dom}\, f_0$. We can choose:

$$\phi_t(x) = p(x) - tq(x)$$

  - for $t \geq 0$, $\phi_t(x)$ is convex in x
  - $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$.

Solving a quasiconvex problem via convex feasibility problems: the idea is to solve the epigraph form of the problem with a sandwich technique in t:

- for fixed t the epigraph form of the original problem reduces to a feasibility convex problem

$$\phi_t(x) \leq 0, \quad f_i(x) \leq 0 \ \forall i, \quad Ax \leq b$$

- if t is too small, the feasibility problem will be infeasible
- if t is too large, the feasibility problem will be feasible

- start with upper and lower bounds on t (termed u and l, resp.) and use a sandwich technique (bisection method): at each iteration use $t = (l+u)/2$ and update the bounds according to the feasibility/infeasibility of the problem.

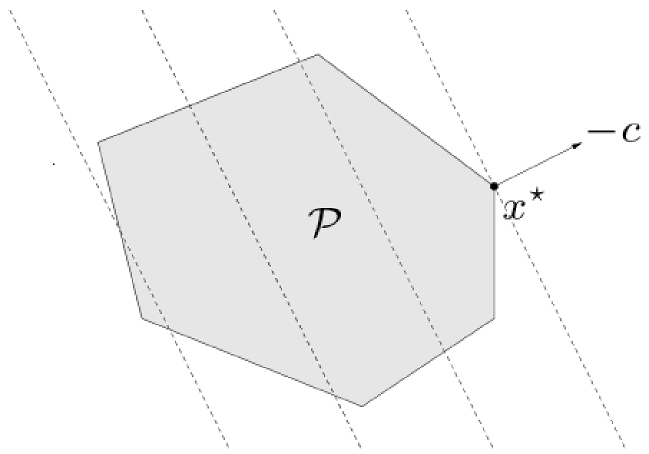
Classes of Convex Problems

Linear Programming (LP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + d \\ \text{subject to} & Gx \le h \\ & Ax = b \end{array}$$

- Convex problem: affine objective and constraint functions.

- Feasible set is a polyhedron:



ℓ_1- and ℓ_∞- Norm Problems as LPs

- ℓ_∞-**norm minimization**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x\|_\infty \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{t,x}{\text{minimize}} & t \\ \text{subject to} & -t\mathbf{1} \leq x \leq t\mathbf{1} \\ & Gx \leq h \\ & Ax = b. \end{array}$$

- ℓ_1-**norm minimization**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|x\|_1 \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

is equivalent to the LP

$$\begin{array}{ll} \underset{t,x}{\text{minimize}} & \sum_i t_i \\ \text{subject to} & -t \leq x \leq t \\ & Gx \leq h \\ & Ax = b. \end{array}$$

Example: Chebyshev Center of a Polyhedron

- The Chebyshev center of a polyhedron $\mathcal{P} = \left\{x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m\right\}$ is the center of the largest inscribed ball $\mathcal{B} = \{x_c + u \mid \|u\| \leq r\}$.

- Let's solve the problem:

$$\begin{array}{ll} \underset{r, x_c}{\text{maximize}} & r \\ \text{subject to} & x \in \mathcal{P} \quad \text{for all} \quad x = x_c + u \mid \|u\| \leq r \end{array}$$

- Observe that $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup_u \left\{a_i^T \left(x_c + u\right) \mid \|u\| \leq r\right\} \leq b_i.$$

- Using Schwartz inequality, the supremum condition can be rewritten as

$$a_i^T x_c + r \|a_i\|_2 \leq b_i.$$

- Hence, the Chebyshev center can be obtained by solving:

$$\begin{array}{ll} \underset{r, x_c}{\text{maximize}} & r \\ \text{subject to} & a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

which is an LP.

Linear-Fractional Programming

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \left(c^T x + d\right) / \left(e^T x + f\right) \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

with $\operatorname{dom} f_0 = \left\{x \mid e^T x + f > 0\right\}$.

- It is a quasiconvex optimization problem (solved by bisection).

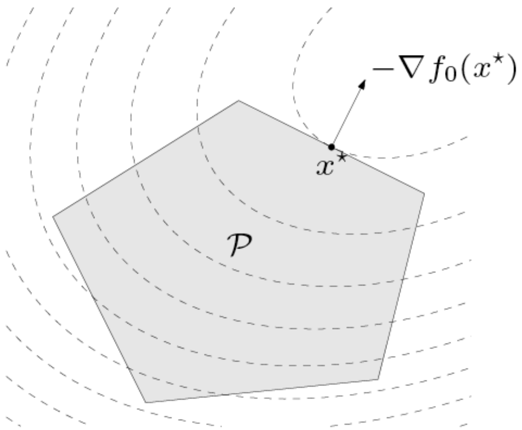
- Interestingly, the following LP is equivalent:

$$\begin{array}{ll} \underset{y,z}{\text{minimize}} & c^T y + dz \\ \text{subject to} & Gy \leq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

Quadratic Programming (QP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (1/2)\, x^T P x + q^T x + r \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- Convex problem (assuming $P \in \mathbf{S}^n \succeq 0$): convex quadratic objective and affine constraint functions.
- Minimization of a convex quadratic function over a polyhedron:



Quadratically Constrained QP (QCQP)

$$\begin{array}{lll} \underset{x}{\text{minimize}} & (1/2)\, x^T P_0 x + q_0^T x + r_0 & \\ \text{subject to} & (1/2)\, x^T P_i x + q_i^T x + r_i \leq 0 & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

- Convex problem (assuming $P_i \in \mathbf{S}^n \succeq 0$): convex quadratic objective and constraint functions.

Second-Order Cone Programming (SOCP)

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f^T x & \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i & i = 1, \ldots, m \\ & Fx = g & \end{array}$$

- Convex problem: linear objective and second-order cone constraints
- For A_i row vector, it reduces to an LP.
- For $c_i = 0$, it reduces to a QCQP.
- More general than QCQP and LP.

Robust LP as an SOCP

- Sometimes, we don't know exactly the parameters of an optimization problem.

- Consider the robust LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \qquad \forall a_i \in \mathcal{E}_i,\ i = 1, \ldots, m \end{array}$$

  where $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\| \leq 1\}$.

- It can be rewritten as the SOCP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \left\| P_i^T x \right\|_2 \leq b_i \qquad i = 1, \ldots, m. \end{array}$$

Generalized Inequality Constraints

- Convex problem with generalized ineq. constraints:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0\left(x\right) \\ \text{subject to} & f_i\left(x\right) \preceq_{K_i} 0 \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  where f_0 is convex and f_i are K_i-convex w.r.t. proper cone K_i.

- It has the same properties as a standard convex problem.

- **Conic form problem**: special case with affine objective and constraints:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

Semidefinite Programming (SDP)

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq G \\ & Ax = b \end{array}$$

- Inequality constraint is called linear matrix inequality (LMI).
- Convex problem: linear objective and linear matrix inequality (LMI) constraints.
- Observe that multiple LMI constraints can always be written as a single one.

- **LP and equivalent SDP**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & \text{diag}\,(Ax - b) \preceq 0 \end{array}$$

- **SOCP and equivalent SDP**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \left[\begin{array}{cc} \left(c_i^T x + d_i\right) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{array} \right] \succeq 0, \quad i = 1, \ldots, m \end{array}$$

- **Eigenvalue minimization**:

$$\underset{x}{\text{minimize}} \quad \lambda_{\max}\left(A\left(x\right)\right)$$

where $A\left(x\right) = A_0 + x_1 A_1 + \cdots + x_n A_n$, is equivalent to SDP

$$\begin{array}{ll} \underset{x}{\text{minimize}} & t \\ \text{subject to} & A\left(x\right) \preceq tI \end{array}$$

- It follows from

$$\lambda_{\max}\left(A\left(x\right)\right) \leq t \iff A\left(x\right) \preceq tI$$

Vector Optimization

- General vector optimization problem:

$$\begin{array}{lll} \underset{x}{\text{minimize}}\ (\text{w.r.t. } K) & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

  where the vector objective $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}^q$ is minimized w.r.t. proper cone $K \subseteq \mathbf{R}^q$.

- Convex vector optimization problem:

$$\begin{array}{lll} \underset{x}{\text{minimize}}\ (\text{w.r.t. } K) & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

  where f_0 is K-convex and $f_1, \ldots, f_m$ are convex.

Pareto Optimality

- Set of achievable objective values:

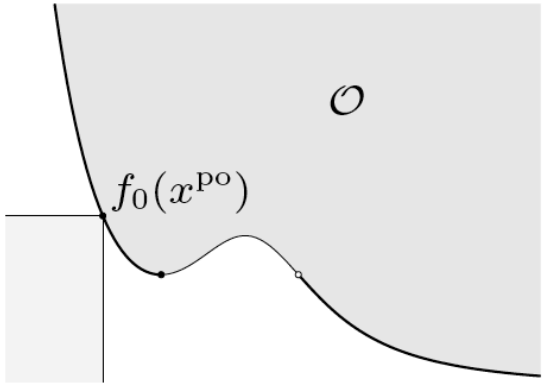
$$\mathcal{O} = \{f_0(x) \mid x \text{ is feasible}\}.$$

- A feasible x is **Pareto optimal** is $f_0(x)$ is a minimal value of $\mathcal{O}$.

- A minimal value x of $\mathcal{O}$ satisfies:

$$y \in \mathcal{O},\ y \preceq_K x \quad \Longrightarrow \quad y = x$$

(in words: x cannot be in the cone of points worse than y)



x^{po} is Pareto optimal

Multicriterion Optimization

- If we now choose the proper cone $K = \mathbf{R}_+^q$ (nonnegative orthant), then the vector optimization becomes a multicriterion optimization with q different objectives:

$$f_0(x) = (F_1(x), \ldots, F_q(x)).$$

- A feasible point x^{po} is Pareto optimal if

$$y \text{ feasible, } f_0(y) \leq f_0(x^{\mathsf{po}}) \quad \Longrightarrow \quad f_0(x^{\mathsf{po}}) = f_0(y).$$

- If there are multiple Pareto optimal values, there is a trade-off among the objectives.

Scalarization for Multicriterion Problems

- To find Pareto optimal points, minimize the positive weighted sum:

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x).$$

- Example: regularized least-squares:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2 \, .$$

Summary

- Thus far, we have seen the basic definitions of convex sets and convex functions with examples and operations that preserve convexity.

- We have then considered convex problems in a variety of forms including quasiconvex problems, vector optimization, Pareto optimality, etc.

- We have also overviewed different classes of convex problems such as LP, QP, QCQP, SOCP, and SDP.

- We can say we have acquired the vocabulary of convex optimization.

References

Chapter 4 of

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

http://www.stanford.edu/~boyd/cvxbook/