Convex Optimization, Game Theory, and Variational Inequality Theory in Multiuser Communication Systems

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Outline of Lecture

- Part I: Overview of Convex Optimization and Game Theory
- Part II: Variational Inequality Theory: The Theory
- Part III: Variational Inequality Theory: Applications

Part I:

Convex Optimization and Game Theory: A Quick Overview

Part I - Outline

- Convex optimization (no need, you are experts already!)
  - The minimum principle
  - Elegant KKT conditions
- Game theory:
  - Nash equilibrium and Pareto optimality
  - Existence/uniqueness theorems
  - Algorithms.

Convex Optimization

Convex Optimization

- By now, my dear students, you should all be extremely familiar (and on your way to becoming experts) with convex optimization.

- For sure you know basic concepts like: convexity, classes of convex problems, Lagrange duality, KKT conditions, relaxation methods, decomposition methods, numerical algorithms, robust worst-case optimization, and a myriad of applications.

- We just need to review the minimum principle and a compact&elegant way of writing the KKT conditions.

Minimum Principle

- For an unconstrained convex optimization problem:

$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x})$$

an optimal solution $\mathbf{x}^{\star}$ must satisfy $\nabla f(\mathbf{x}^{\star}) = 0$.

- The extension of this optimality criterion to a constrained convex optimization problem

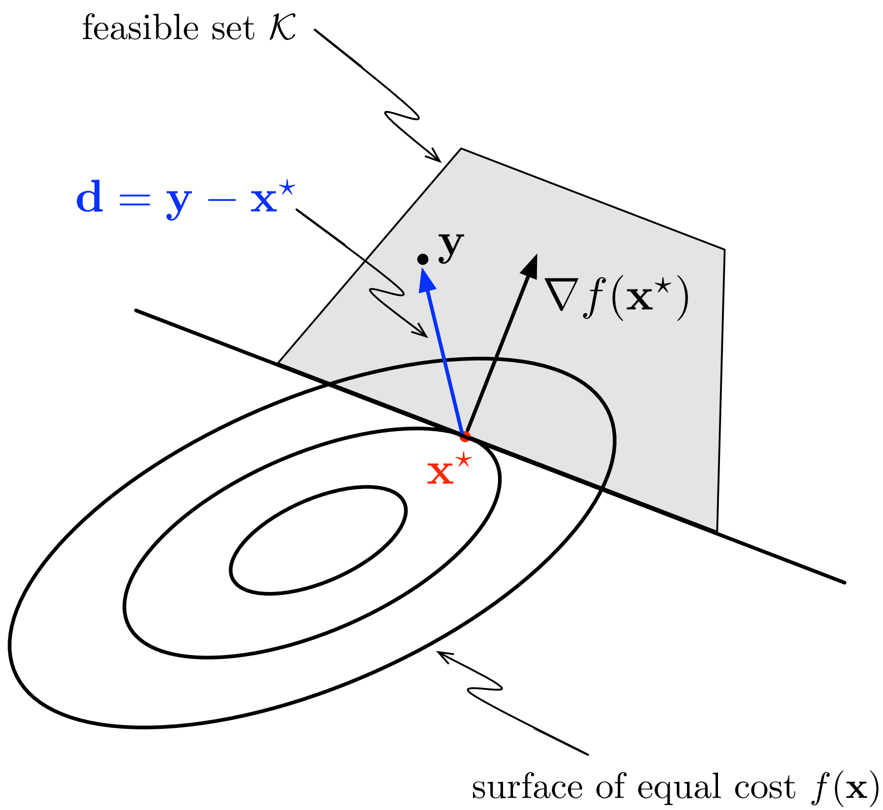
$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{K} \end{array}$$

is the so-called **minimum principle**:

$$(\mathbf{y} - \mathbf{x}^{\star})^{T} \nabla f(\mathbf{x}^{\star}) \geq 0 \quad \forall \mathbf{y} \in \mathcal{K}.$$

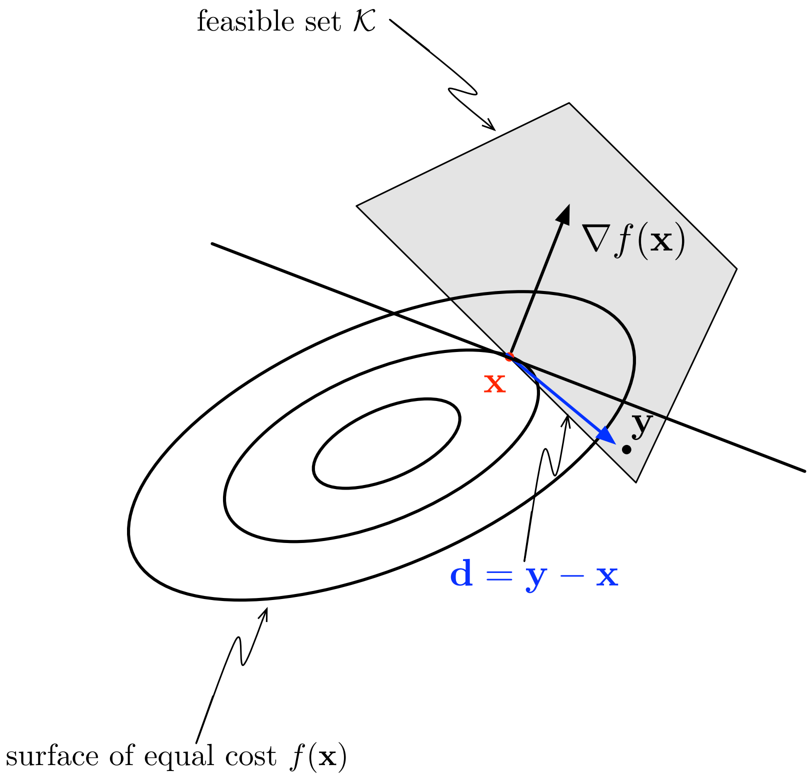
Geometrical Interpretation of the Minimum Principle

- A feasible $\mathbf{x}^{\star}$ that satisfies the minumum principle: $\nabla f(\mathbf{x}^{\star})$ forms a nonobtuse angle with all feasible vectors $\mathbf{d} = \mathbf{y} - \mathbf{x}^{\star}$ emanating from $\mathbf{x}^{\star}$:



Geometrical Interpretation of the Minimum Principle

- A feasible $\mathbf{x}^{\star}$ that does NOT satisfy the minumum principle: there are other feasible points $\mathbf{y} \neq \mathbf{x}^{\star}$ such that $f(\mathbf{y}) < f(\mathbf{x}^{\star})$:



Lagrangian

- Consider an optimization problem in standard form (not necessarily convex)

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

  with variable $x \in \mathbf{R}^n$, domain $\mathcal{D}$, and optimal value $p^\star$.

- The *Lagrangian* is a function $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\operatorname{dom} L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$, defined as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  where λ_i is the Lagrange multiplier associated with $f_i(x) \le 0$ and ν_i is the Lagrange multiplier associated with $h_i(x) = 0$.

Karush-Kuhn-Tucker (KKT) Conditions

KKT conditions (for differentiable f_i, h_i):

1. primal feasibility: $f_i(x^\star) \leq 0,\ i = 1, \ldots, m,\ h_i(x^\star) = 0,\ i = 1, \ldots, p$

2. dual feasibility: $\lambda^\star \geq 0$

3. complementary slackness: $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$

4. zero gradient of Lagrangian with respect to x:

$$\nabla f_0(x^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla f_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star \nabla h_i(x^\star) = 0$$

Elegant and Compact KKT Conditions

- We can make use of the compact notation $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a}^T \mathbf{b} = 0$ to write the primal-dual feasibility and complementary slackness KKT conditions as

$$0 \leq \lambda_i^\star \perp - f_i \left(\mathbf{x}^\star \right) \geq 0 \quad \text{for } i = 1, \ldots, m.$$

- By stacking the f_i's, h_i's, and $\lambda_i^\star$'s in vectors, we can finally write the KKT conditions in the following compact form:

$$\begin{aligned} \mathbf{0} \leq \boldsymbol{\lambda}^\star \perp - \mathbf{f} \left(\mathbf{x}^\star \right) &\geq \mathbf{0} \\ \mathbf{h} \left(\mathbf{x}^\star \right) &= \mathbf{0} \end{aligned}$$

and

$$\nabla f_0 \left(\mathbf{x}^\star \right) + \nabla \mathbf{f}^T \left(\mathbf{x}^\star \right) \boldsymbol{\lambda}_i^\star + \nabla \mathbf{h}^T \left(\mathbf{x}^\star \right) \boldsymbol{\nu}_i^\star = \mathbf{0}.$$

Game Theory

Game Theory and Nash Equilibrium

- Game theory deals with problems with interacting decision-makers (called players).

- Noncooperative game theory is a branch of game theory for the resolution of conflicts among selffish players, each of which tries to optimize his own objective function.

- A solution to such a competitive game is an equilibrium point where each user is unilaterally happy and does not want to deviate: it is the so-called **Nash Equilibrium** (NE).

- We will consider two important types of noncooperative games: Nash Equilibrium Problems (NEP) and Generalized NEP (GNEP).

Nash Equilibrium Problems (NEP)

- Mathematically, we can define a NEP as a set of *coupled* optimization problems:

$$(\mathcal{G}): \qquad \begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i \end{array} \qquad i = 1, \ldots, Q$$

where:

  - $\mathbf{x}_i \in \mathbb{R}^{n_i}$ is the strategy of player i
  - $\mathbf{x}_{-i} \triangleq (\mathbf{x}_j)_{j \neq i}$ are the strategies of all the players except i
  - $\mathcal{K}_i \subseteq \mathbb{R}^{n_i}$ is the strategy set of player i
  - $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is the cost function of player i.

- How to define a solution of the game?

- Solution of $\mathcal{G}$: A (pure strategy) Nash Equilibrium (NE) is a feasible $\mathbf{x}^\star = (\mathbf{x}_i^\star)_{i=1}^Q$ such that

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \leq f_i(\mathbf{y}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{y}_i \in \mathcal{K}_i, \quad \forall i = 1, \ldots, Q$$

  A NE is a strategy profile where every player is *unilaterally* happy.

- Life is not so easy:
  - A (pure strategy) NE may not exist or be unique
  - Even when the NE is unique, there is no guarantee of convergence of iterative (best-response) algorithms.

- How to analyze a game?

- How to design convergent distributed algorithms?

Minimum Principle for NEPs

- A NE is defined as a simultaneous solution of each of the single-player optimization problems: given the other players $\mathbf{x}_{-i}^{\star}$, $\mathbf{x}_i^{\star}$ must be the solution to

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}^{\star}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i. \end{array}$$

- The minimum principle for a convex game is then, for each $i = 1, \ldots, Q$:

$$(\mathbf{y}_i - \mathbf{x}_i^{\star})^T \nabla f_i \left(\mathbf{x}_i^{\star}, \mathbf{x}_{-i}^{\star}\right) \geq 0 \quad \forall \mathbf{y}_i \in \mathcal{K}_i.$$

KKT Conditions for NEPs

- Suppose now that each set $\mathcal{K}_i$ is described by a set of equalities and inequalities:

$$\mathcal{K}_i = \{\mathbf{x}_i : \mathbf{g}_i(\mathbf{x}_i) \leq \mathbf{0},\ \mathbf{h}_i(\mathbf{x}_i) = \mathbf{0}\}.$$

- Then we can write the coupled KKT conditions for the convex NEP, for each $i = 1, \ldots, Q$, as

$$\begin{aligned} \mathbf{0} \leq \boldsymbol{\lambda}_i^\star \perp -\mathbf{g}_i(\mathbf{x}^\star) &\geq \mathbf{0} \\ \mathbf{h}_i(\mathbf{x}_i) &= \mathbf{0} \end{aligned}$$

and

$$\nabla f_i(\mathbf{x}^\star) + \nabla \mathbf{g}_i^T(\mathbf{x}_i^\star)\boldsymbol{\lambda}_i^\star + \nabla \mathbf{h}_i^T(\mathbf{x}_i^\star)\boldsymbol{\nu}_i^\star = \mathbf{0}.$$

Pareto Optimality

- Pareto optimality is the natural extension of the concept of optimality in classical optimization to multi-objective optimization.

- A point $\mathbf{x}$ is Pareto optimal if its set of objectives $(f_1(\mathbf{x}), \ldots, f_Q(\mathbf{x}))$ cannot be improved (element by element) by another point.

- Note that some objectives of a Pareto optimal point may be improved by another point, but not all the objectives at the same time.

- Therefore, a multi-objective optimization problem has associated a Pareto-optimal frontier rather than an optimal point.

- In the context of game theory, one may formulate the multi-objective (possibly nonconvex) formulation of the game as

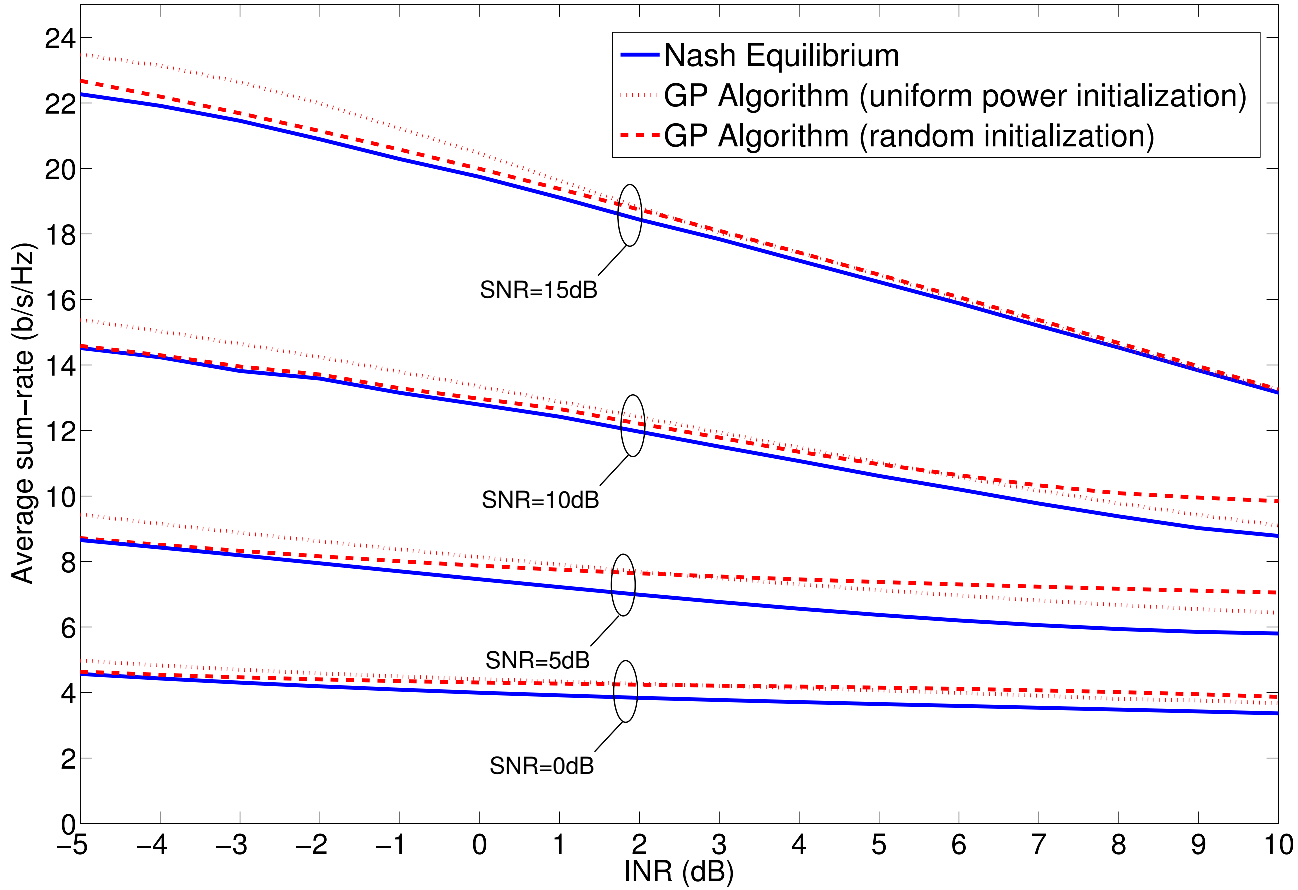
$$\begin{array}{ll} \underset{\mathbf{x}_1,\ldots,\mathbf{x}_Q}{\text{minimize}} & \sum_{i=1}^{Q} w_i f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i \quad \forall i \end{array}$$

  where the weights w_i explore (the convex hull of) the Pareto optimality frontier. One common choice is to choose equal weights $w_i = 1$, obtaining the sum-performance (social optimum).

- We can then compare the performance achieved by a NE versus that achieved by a Pareto optimal point in terms of sum-performance. This is related to the concept of price of anarchy in game theory.

- The loss of performance can be large in some cases. In most practical cases, however, does not seem to be too significant.

• As an example, we plot the sum-performance (sum-rate) achieved by the NE as well as the sum-Pareto-optimal point in an adhoc wireless multiuser system:



Solution Analysis: Existence of NE

- **Theorem. [(existence theorem) Debreu-Fan-Glicksberg (1952)]** *Given the game $\mathcal{G} = <\mathcal{K}, \mathbf{f}>$, suppose that:*
  - *The action space $\mathcal{K}$ is compact and convex;*
  - *The cost-functions $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ are continuous in $\mathbf{x} \in \mathcal{K}$ and quasi-convex in $\mathbf{x}_i \in \mathcal{K}_i$, for any given $\mathbf{x}_{-i}$.*

  Then, there exists a pure strategy NE.

- Existence may also follow from the special structure of the game, for example:
  - Potential games
  - Supermodular games.

Solution Analysis: Uniqueness of NE for Convex Games

- **Theorem. [Rosen uniqueness theorem (1965)]** *Given the game* $\mathcal{G} = <\mathcal{K}, \mathbf{f}>$*, suppose that:*
  - *The action space* $\mathcal{K}$ *is compact and convex;*
  - *The cost-functions* $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ *are continuous in* $\mathbf{x} \in \mathcal{K}$ *and quasi-convex in* $\mathbf{x}_i \in \mathcal{K}_i$*, for any given* $\mathbf{x}_{-i}$*;*
  - *The* Diagonal Strict Convexity *(DSC) property holds true: given* $\mathbf{g}(\mathbf{x}, \mathbf{r}) \triangleq (r_i \nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$*,*

  $$\text{DSC: } \exists \mathbf{r} > \mathbf{0} \, : \, (\mathbf{x} - \mathbf{y})^T (\mathbf{g}(\mathbf{x}, \mathbf{r}) - \mathbf{g}(\mathbf{y}, \mathbf{r})) > 0, \; \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}, \, \mathbf{x} \neq \mathbf{y}$$

  Then, there exists a unique pure strategy NE.

- The DSC property is not easy to check and may be too restrictive (we want the conditions to be satisfied for *all* $\mathbf{x} \in \mathcal{K}$).

Solution Analysis: Uniqueness of NE for Standard Games

- Let define the best-respose $\mathcal{B}_i(\mathbf{x}_{-i})$ of each player i as the set of optimal solutions of player i's optimization problem for any given $\mathbf{x}_{-i}$:

$$\mathcal{B}_i(\mathbf{x}_{-i}) \triangleq \{\mathbf{x}_i \in \mathcal{K}_i \, : \, f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \leq f_i(\mathbf{y}_i, \mathbf{x}_{-i}), \quad \forall \mathbf{y}_i \in \mathcal{K}_i\} \, .$$

- **Standard function [Yates, 1995]**: A function $\mathbf{g} : \mathcal{K} \rightarrow \mathbb{R}_+^m$ is said to be standard if it has the two following properties:
  - *Monotonicity*: $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}, \quad \mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{g}(\mathbf{x}) \leq \mathbf{g}(\mathbf{y})$ (component-wise)
  - *Scalability*: $\forall \alpha > 0, \; \forall \mathbf{x} \in \mathcal{K}, \quad \mathbf{g}(\alpha \, \mathbf{x}) \leq \alpha \, \mathbf{g}(\mathbf{x})$.

- If the best-response map $\mathcal{B}(\mathbf{x}) = (\mathcal{B}_i(\mathbf{x}_{-i}))_{i=1}^{Q}$ is a standard function (requiring $\mathcal{B}_i(\mathbf{x}_{-i})$ be a single-valued function), then the NE of the game $\mathcal{G} =< \mathcal{K}, \mathbf{f} >$ is unique.

- The above requirement is quite strong and in general is not satisfied in practice by games of our interest.

Algorithms for Achieving NE

- Best-response (BR) based dynamics:
  - Convergence requires contraction or monotonicity of the BR map;
  - Contraction can be studied using classical fixed-point theory if the BR is known in closed form;
  - Monotonicity is guaranteed if the BR is a standard function (too restrictive).

- ODE approximation: the idea is to introduce a time-continuous ordinary differential equation (ODE), whose equilibrium coincides with the NE of the game:
  - Convergence to a NE is guaranteed under the globally asymptotically stability (GAS) of the equilibrium of the ODE;
  - Sufficient conditions for the DSC implies the GAS.

Summary

- You already knew the basic concepts in convex optimization theory.
- We have reviewed the minimum principle and the KKT conditions.
- Within the context of game theory, we have defined the concepts of NE, Pareto optimality, and have formulated NEPs.
- We have given the state of the art for the solution analysis of NEPs as well as practical algorithms.

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Part II:

Variational Inequality Theory: The Theory

Part II - Outline

- VI as a general framework
- Alternative formulations of VI: KKT conditions, primal-dual form
- Solution analysis of VI: existence, uniqueness, solution as a projection
- Algorithms for VI
- NEP as a VI: solution analysis and algorithms
- GNEP as a VI: variational equilibria and algorithms

VI as a General Framework

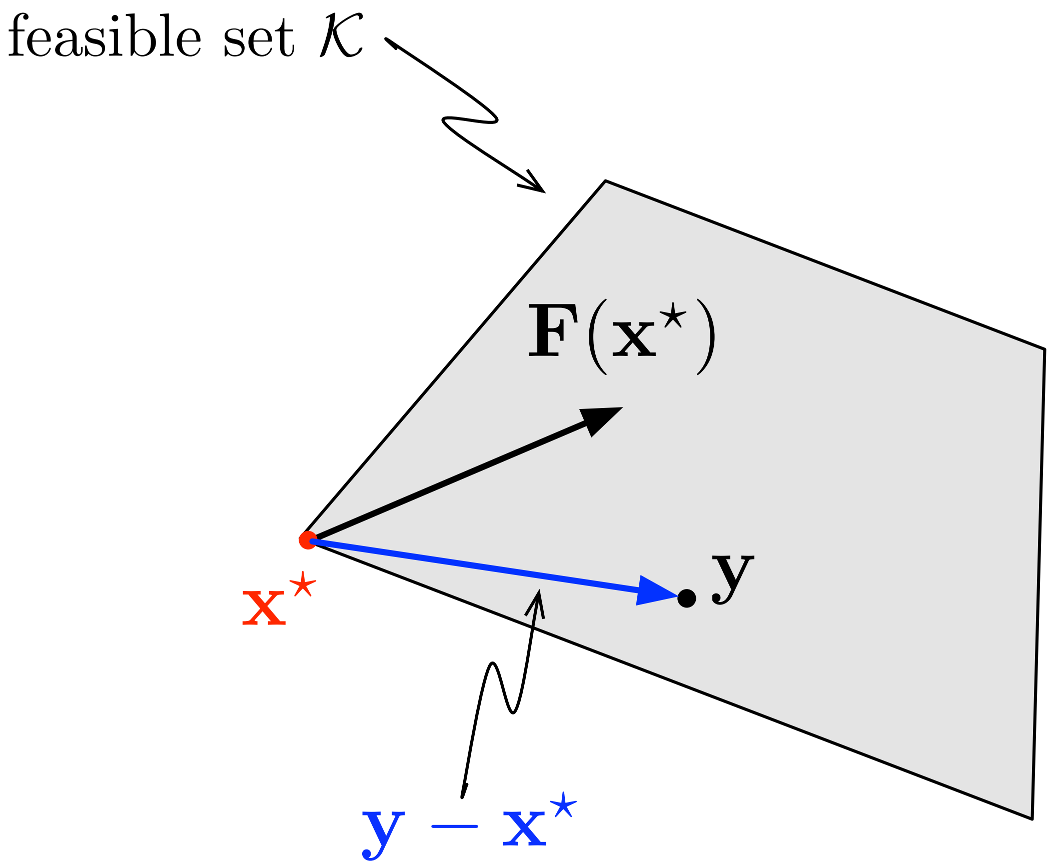
The VI Problem

Given a set $\mathcal{K} \subseteq \mathbb{R}^n$ and a mapping $\mathbf{F} : \mathcal{K} \rightarrow \mathbb{R}^n$, the VI problem VI$(\mathcal{K}, \mathbf{F})$ is to find a vector $\mathbf{x}^\star \in \mathcal{K}$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) \geq 0 \qquad \forall \mathbf{y} \in \mathcal{K}.$$

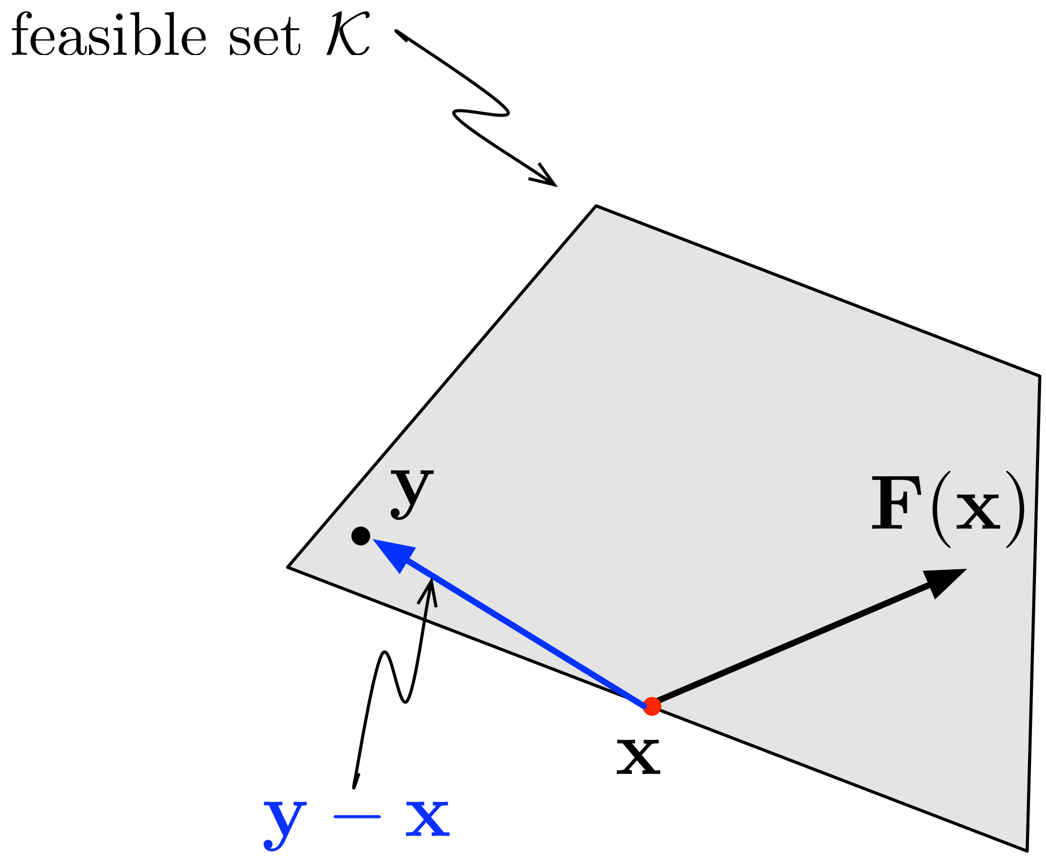
Geometrical Interpretation of the VI

- A feasible point $\mathbf{x}^\star$ that is a solution of the VI$(\mathcal{K}, \mathbf{F})$: $\mathbf{F}(\mathbf{x}^\star)$ forms an acute angle with all the feasible vectors $\mathbf{y} - \mathbf{x}^\star$



Geometrical Interpretation of the VI

- A feasible point $\mathbf{x}^{\star}$ that is NOT a solution of the VI$(\mathcal{K}, \mathbf{F})$



Convex Optimization as a VI

- Convex optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{K} \end{array}$$

  where $\mathcal{K} \subseteq \mathbb{R}^n$ is a convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

- Minimum principle: The problem above is equivalent to finding a point $\mathbf{x}^\star \in \mathcal{K}$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \nabla f(\mathbf{x}^\star) \geq 0 \quad \forall \mathbf{y} \in \mathcal{K} \quad \Longleftrightarrow \quad \text{VI}(\mathcal{K}, \nabla f)$$

  which is a special case of VI with $\mathbf{F} = \nabla f$.

- It seems that a VI is more general than a convex optimization problem only when $\mathbf{F} \neq \nabla f$.

- But is it really that significative? The answer is affirmative.

- The VI$(\mathcal{K}, \mathbf{F})$ encompasses a wider range of problems than classical optimization whenever $\mathbf{F} \neq \nabla f$ ($\Leftrightarrow$ $\mathbf{F}$ has not a symmetric Jacobian).

- Some examples of relevant problems that can be cast as a VI include NEPs, GNEPs, system of equations, nonlinear complementary problems, fixed-point problems, saddle-point problems, etc.

NEP as a VI

- The minimum principle for a convex game is, for each $i = 1, \ldots, Q$:

$$\left(\mathbf{y}_i - \mathbf{x}_i^\star\right)^T \nabla f_i \left(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star\right) \geq 0 \quad \forall \mathbf{y}_i \in \mathcal{K}_i.$$

- We can write it in a more compact way as

$$\left(\mathbf{y} - \mathbf{x}^\star\right)^T \mathbf{F}\left(\mathbf{x}^\star\right) \geq 0 \quad \forall \mathbf{y} \in \mathcal{K}$$

by defining $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_Q$ and $\mathbf{F} = \left(\nabla_{\mathbf{x}_i} f_i\right)_{i=1}^Q$ (note that $\mathbf{F} \neq \nabla f$).

- Hence, the interpretation of the NEP as a VI:

$$\min_{\mathbf{x}_i \in \mathcal{K}_i} f_i(\mathbf{x}_i, \mathbf{x}_{-i}), \quad \forall i \ = 1, \ldots, Q \qquad \Longleftrightarrow \qquad \text{VI}(\mathcal{K}, \mathbf{F}).$$

System of Equations as a VI

- In some engineering problems, we may not want to minimize a function but instead finding a solution to a system of equations:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

- This can be cast as a VI by choosing $\mathcal{K} = \mathbb{R}^n$.

- Hence,

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \qquad \Longleftrightarrow \qquad \text{VI}(\mathbb{R}^n, \mathbf{F}).$$

Nonlinear Complementary Problem (NCP) as a VI

- The NCP is a unifying mathematical framework that includes linear programming, quadratic programming, and bi-matrix games.

- The NCP($\mathbf{F}$) is to find a vector $\mathbf{x}^\star$ such that

$$\text{NCP}(\mathbf{F}): \qquad \mathbf{0} \le \mathbf{x}^\star \perp \mathbf{F}(\mathbf{x}^\star) \ge \mathbf{0}.$$

- An NCP can be cast as a VI by choosing $\mathcal{K} = \mathbb{R}^n_+$:

$$\text{NCP}(\mathbf{F}) \qquad \Longleftrightarrow \qquad \text{VI}(\mathbb{R}^n_+, \mathbf{F}).$$

Linear Complementary Problem (LCP) as a VI

- An LCP is just an NCP with an affine function: $\mathbf{F}(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{q}$:

$$\text{LCP}(\mathbf{M},\, \mathbf{q}) : \qquad \mathbf{0} \,\leq\, \mathbf{x}^{\star} \,\perp\, \mathbf{M}\mathbf{x}^{\star} + \mathbf{q} \,\geq \mathbf{0}.$$

- An LCP can be cast as a VI by choosing $\mathcal{K} = \mathbb{R}^n_+$:

$$\text{LCP}(\mathbf{M},\, \mathbf{q}) \qquad \Longleftrightarrow \qquad \text{VI}(\mathbb{R}^n_+, \mathbf{M}\mathbf{x} + \mathbf{q}).$$

- Examples of LCP: Quadratic (linear) programming, Canonical Bi-matrix games, Nash-Cournot games.

Quadratic (Linear) Programming as an LCP

- Consider the following QP (non necessarily convex):

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{c}^T\mathbf{x} + r \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

  where $\mathbf{P} \in \mathbb{R}^{n\times n}$ is *symmetric* (the case $\mathbf{P} = \mathbf{0}$ gives rise to an LP), $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $r \in \mathbb{R}$.

- The KKT are necessary conditions for the optimality (and sufficient if $\mathbf{P} \succeq \mathbf{0}$)

$$\begin{array}{lll} \boldsymbol{\lambda} = \mathbf{P}\mathbf{x} + \mathbf{c} + \mathbf{A}^T\boldsymbol{\mu} \geq \mathbf{0}, & \mathbf{x} \geq \mathbf{0}, & \boldsymbol{\lambda}^T\mathbf{x} = 0, \\ \mathbf{b} - \mathbf{A}\mathbf{x} \geq \mathbf{0} & \boldsymbol{\mu} \geq \mathbf{0} & \boldsymbol{\mu}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0 \end{array}$$

- The KKT conditions can be written more compactly as

$$\begin{aligned} \mathbf{0} \leq \mathbf{x} \quad &\perp \quad \mathbf{P}\mathbf{x} + \mathbf{A}^T \boldsymbol{\mu} + \mathbf{c} \geq \mathbf{0} \\ \mathbf{0} \leq \boldsymbol{\mu} \quad &\perp \quad -\mathbf{A}\mathbf{x} + \mathbf{b} \geq \mathbf{0} \end{aligned}$$

which can be rewritten as LCP$(\mathbf{M}, \mathbf{q})$

$$\mathbf{0} \leq \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\mu} \end{bmatrix} \perp \underbrace{\begin{bmatrix} \mathbf{P} & \mathbf{A}^T \\ -\mathbf{A} & 0 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\mu} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{c} \\ \boldsymbol{b} \end{bmatrix}}_{\mathbf{q}} \geq \mathbf{0}$$

Fixed-Point Problem as a VI

- In other engineering problems, we may need to find the (unconstrained) fixed-point of a mapping $\mathbf{G}(\mathbf{x})$:

$$\text{find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} = \mathbf{G}(\mathbf{x}) .$$

- This can be easily cast as a VI by noticing that a fixed-point of $\mathbf{G}(\mathbf{x})$ corresponds to a solution to a system of equations with function $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \mathbf{G}(\mathbf{x})$:

$$\mathbf{x} = \mathbf{G}(\mathbf{x}) \qquad \Longleftrightarrow \qquad \text{VI}(\mathbb{R}^n, \mathbf{I} - \mathbf{G}).$$

- Similarly, for *constrained* fixed-point equations:

$$\text{find } \mathbf{x} \in \mathcal{K} \text{ such that } \mathbf{x} = \mathbf{G}(\mathbf{x}) \qquad \Longleftrightarrow \qquad \text{VI}(\mathcal{K}, \mathbf{I} - \mathbf{G}).$$

Saddle-Point Problem as a VI

- Given two sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$, and a function $\mathcal{L}(\mathbf{x}, \mathbf{y})$, the saddle point problem is to find a pair $(\mathbf{x}^\star, \mathbf{y}^\star) \in \mathcal{X} \times \mathcal{Y}$ such that

$$\mathcal{L}(\mathbf{x}^\star, \mathbf{y}) \leq \mathcal{L}(\mathbf{x}^\star, \mathbf{y}^\star) \leq \mathcal{L}(\mathbf{x}, \mathbf{y}^\star) \quad \forall (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$$

  or, equivalently,

$$\mathcal{L}(\mathbf{x}^\star, \mathbf{y}^\star) = \min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{y} \in \mathcal{Y}} \mathcal{L}(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{y} \in \mathcal{Y}} \min_{\mathbf{x} \in \mathcal{X}} \mathcal{L}(\mathbf{x}, \mathbf{y}).$$

- Assuming $\mathcal{L}(\mathbf{x}, \mathbf{y})$ continuously differentiable and "convex-concave", if the saddle-point exists, then

$$\begin{array}{ll} (\mathbf{x} - \mathbf{x}^\star)^T \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^\star, \mathbf{y}^\star) \geq 0 & \forall \mathbf{x} \in \mathcal{X} \\ (\mathbf{y} - \mathbf{y}^\star)^T (-\nabla_{\mathbf{y}} \mathcal{L}(\mathbf{x}^\star, \mathbf{y}^\star)) \geq 0 & \forall \mathbf{y} \in \mathcal{Y} \end{array}$$

which is equivalent to the VI$(\mathcal{K}, \mathbf{F})$ with

$$\mathcal{K} = \mathcal{X} \times \mathcal{Y} \qquad \text{and} \qquad \mathbf{F} = \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \end{bmatrix}.$$

Alternative Formulations of VI: KKT Conditions

- Suppose that the (convex) feasible set $\mathcal{K}$ of VI$(\mathcal{K}, \mathbf{F})$ is described by a set of inequalities and equalities

$$\mathcal{K} = \{\mathbf{x} : \mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}\}.$$

- The KKT conditions of VI$(\mathcal{K}, \mathbf{F})$ are

$$\begin{aligned}
\mathbf{0} &= \mathbf{F}(\mathbf{x}) + \nabla \mathbf{g}^T(\mathbf{x})\,\boldsymbol{\lambda} + \nabla \mathbf{h}^T(\mathbf{x})\,\boldsymbol{\nu} \\
\mathbf{0} &\leq \boldsymbol{\lambda} \perp \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\
\mathbf{0} &= \mathbf{h}(\mathbf{x}).
\end{aligned}$$

- To derive the KKT conditions it suffices to realize that if $\mathbf{x}$ is a solution to VI$(\mathcal{K}, \mathbf{F})$ then it must solve the following convex optimization problem and vice-versa:

$$\begin{array}{ll} \underset{\mathbf{y}}{\text{minimize}} & \mathbf{y}^T \mathbf{F}\left(\mathbf{x}^\star\right) \\ \text{subject to} & \mathbf{y} \in \mathcal{K}. \end{array}$$

(Otherwise, there would be a point $\mathbf{y}$ with $\mathbf{y}^T \mathbf{F}\left(\mathbf{x}^\star\right) < \mathbf{x}^{\star T} \mathbf{F}\left(\mathbf{x}^\star\right)$ which would imply $\left(\mathbf{y} - \mathbf{x}^\star\right)^T \mathbf{F}\left(\mathbf{x}^\star\right) < 0.$)

- The KKT conditions of the VI follow from the KKT conditions of this problem noting that the gradient of the objective is $\mathbf{F}\left(\mathbf{x}^\star\right)$.

Alternative Formulations of VI: Primal-Dual Representation

- We can now capitalize on the KKT conditions of VI$(\mathcal{K}, \mathbf{F})$ to derive an alternative representation of the VI involving not only the primal variable $\mathbf{x}$ but also the dual variables $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$.

- Consider VI$(\tilde{\mathcal{K}}, \tilde{\mathbf{F}})$ with $\tilde{\mathcal{K}} = \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p$ and

$$\tilde{\mathbf{F}}\left(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}\right) = \left[\begin{array}{c} \mathbf{F}(\mathbf{x}) + \nabla \mathbf{g}^T\left(\mathbf{x}\right)\boldsymbol{\lambda} + \nabla \mathbf{h}^T\left(\mathbf{x}\right)\boldsymbol{\nu} \\ -\mathbf{g}\left(\mathbf{x}\right) \\ \mathbf{h}\left(\mathbf{x}\right) \end{array}\right].$$

- The KKT conditions of VI$(\tilde{\mathcal{K}}, \tilde{\mathbf{F}})$ coincide with those of VI$(\mathcal{K}, \mathbf{F})$. Hence, both VIs are equivalent.

- VI$(\mathcal{K}, \mathbf{F})$ is the original (primal) representation whereas VI$(\tilde{\mathcal{K}}, \tilde{\mathbf{F}})$ is the so-called primal-dual form as it makes explicit both primal and dual variables.

- In fact, this primal-dual form is the VI representation of the KKT conditions of the original VI.

Generalization of VI: Quasi-VI

- The QVI is an extension of a VI in which the defining set of the problem varies with the variable.

- Let $\mathcal{K} : \mathbb{R}^n \ni \mathbf{x} \rightrightarrows \mathcal{K}(\mathbf{x}) \subseteq \mathbb{R}^n$ be a point-to-set mapping from $\mathbb{R}^n$ into subsets of $\mathbb{R}^n$; let $\mathbf{F} : \mathbb{R}^n \ni \mathbf{x} \to \mathbb{R}^n$ be a vector function on $\mathbb{R}^n$. The QVI defined by the pair $(\mathcal{K}, \mathbf{F})$ is to find a vector $\mathbf{x}^\star \in \mathbf{K}(\mathbf{x}^\star)$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) \geq 0, \quad \forall \mathbf{y} \in \mathcal{K}(\mathbf{x}^\star)$$

- If $\mathcal{K}(\mathbf{x}) = \mathcal{K} \subseteq \mathbb{R}^n$, the QVI reduces to the VI

- If $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, the QVI reduces to the classical fixed-point problem of the point-to-set map $\mathcal{K}(\mathbf{x})$: find a $\mathbf{x}$ such that $\mathbf{x} \in \mathbf{K}(\mathbf{x})$.

Solution Analysis of VI

Existence of a Solution

- **Theorem.** *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be compact and convex, and let $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$ be continuous. Then, the VI$(\mathcal{K}, \mathbf{F})$ has a nonempty and compact solution set.*

- Without boundedness of $\mathcal{K}$, the existence of a solution needs some additional properties on the vector function $\mathbf{F}$ (as for convex optimization problems):
  - "Coercivity"
  - Monotonicity/P-properties.

Monotonicity Properties of $\mathbf{F}$: Outline

- Monotonicity properties of vector functions
- Convex programming - a special case: monotonicity properties are satisfied immediately by gradient maps of convex functions
- In a sense, role of monotonicity in VIs is similar to that of convexity in optimization
- Existence (uniqueness) of solutions of VIs and convexity of solution sets under monotonicity properties

Monotonicity Properties of $\mathbf{F}$ (I)

- A mapping $\mathbf{F} : \mathcal{K} \rightarrow \mathbb{R}^n$ is said to be

(i) *monotone* on $\mathcal{K}$ if

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) \geq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$$

(ii) *strictly monotone* on $\mathcal{K}$ if

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) > 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K} \text{ and } \mathbf{x} \neq \mathbf{y}$$

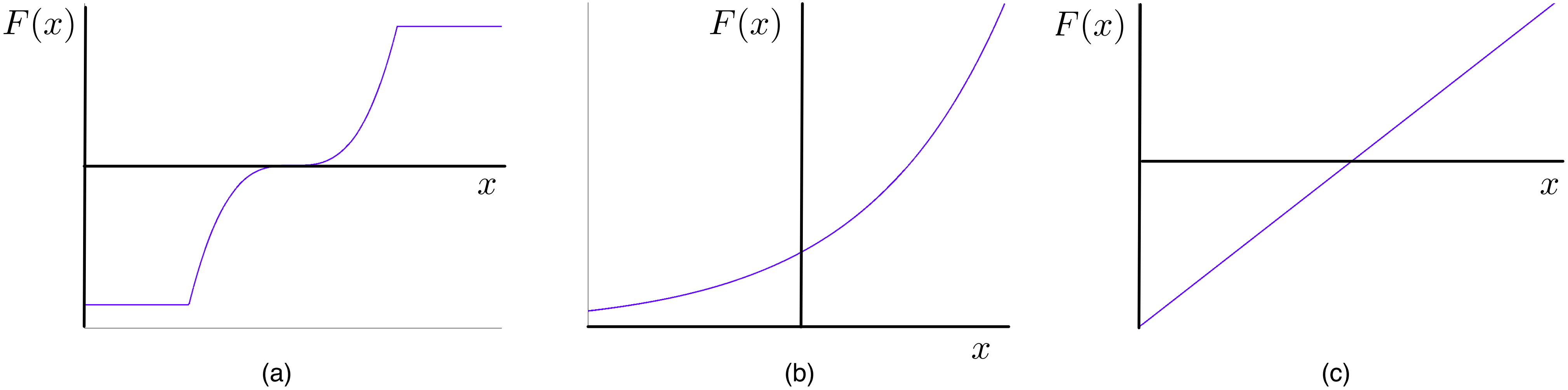
(iii) *strongly* monotone on $\mathcal{Q}$ if there exists constant $c_{\mathsf{sm}} > 0$ such that

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) \geq c_{\mathsf{sm}} \| \mathbf{x} - \mathbf{y} \|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$$

The constant c_{sm} is called strong monotonicity constant

Monotonicity Properties of $\mathbf{F}$ (II)

- Example of (a) monotone, (b) strictly monotone, and (c) strongly monotone functions:



Monotonicity Properties of $\mathbf{F}$ (III)

- Relations among the monotonicity properties and connection with the positive semidefinitess of the Jacobian matrix $J\mathbf{F}(\mathbf{x})$ of $\mathbf{F}$:

$$\begin{array}{ccccc} \text{strongly monotone} & \Rightarrow & \text{strictly monotone} & \Rightarrow & \text{monotone} \\ \Updownarrow & & \Uparrow & & \Updownarrow \\ J\mathbf{F}(\mathbf{x}) - c\,\mathbf{I} \succeq \mathbf{0} & \Rightarrow & J\mathbf{F}(\mathbf{x}) \succ \mathbf{0} & \Rightarrow & J\mathbf{F}(\mathbf{x}) \succeq \mathbf{0} \end{array}$$

- For an affine map $\mathbf{F} = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ (not necessarily symmetric) and $\mathbf{b} \in \mathbb{R}^n$, we have:

$$\begin{array}{l} \text{strongly monotone} \Leftrightarrow \text{strictly monotone} \Leftrightarrow \mathbf{A} \succ \mathbf{0} \\ \qquad \Rightarrow \text{monotone} \Leftrightarrow \mathbf{A} \succeq \mathbf{0} \end{array}$$

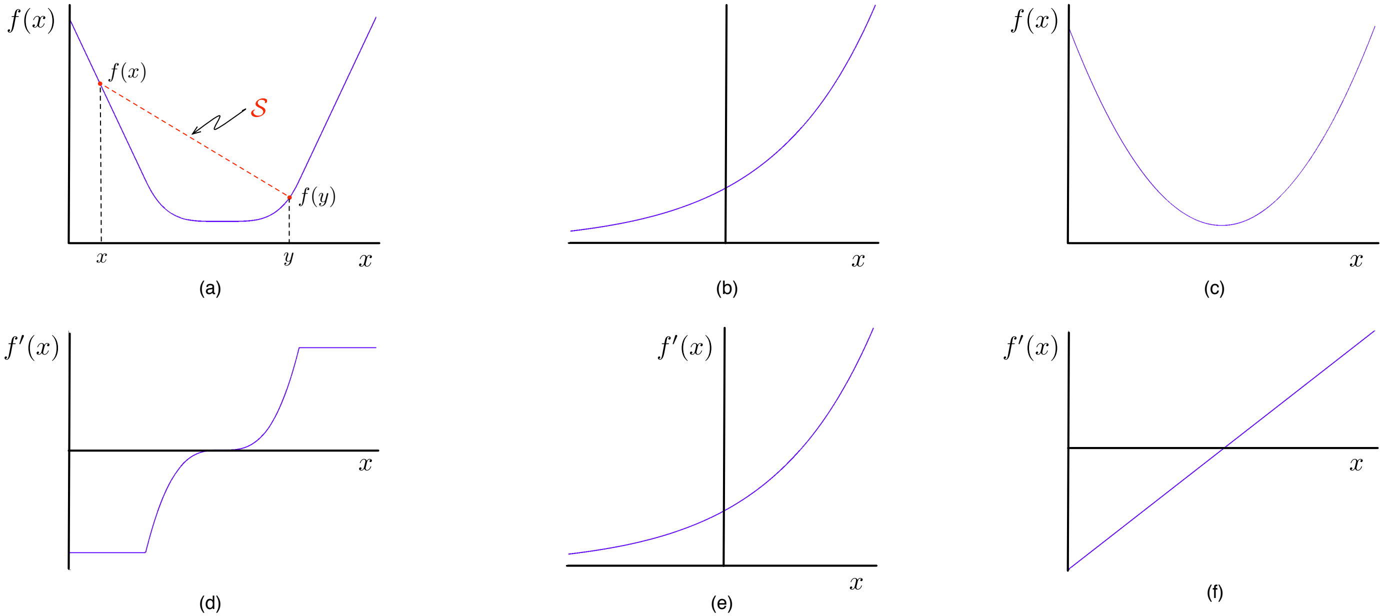
Monotonicity Properties of $\mathbf{F}$ (IV)

- If $\mathbf{F} = \nabla f$, the monotonicity properties can be related to the convexity properties of f

a) f convex $\Leftrightarrow$ ∇f monotone $\Leftrightarrow$ $\nabla^2 f \succeq \mathbf{0}$

b) f strictly convex $\Leftrightarrow$ ∇f strictly monotone $\Leftarrow$ $\nabla^2 f \succ \mathbf{0}$

c) f strongly convex $\Leftrightarrow$ ∇f strongly monotone $\Leftrightarrow$ $\nabla^2 f - c\,\mathbf{I} \succeq \mathbf{0}$



Why Are Monotone Mappings Important?

- Arise from important classes of optimization/game-theoretic problems.

- Can articulate existence/uniqueness statements for such problems and VIs.

- Convergence properties of algorithms may sometimes (but not always) be restricted to such monotone problems.

Solution Analysis Under Monotonicity

Theorem. *Let* $\mathcal{K} \subseteq \mathbb{R}^n$ *be a convex closed set, and let* $\mathbf{F} : \mathcal{K} \rightarrow \mathbb{R}^n$ *be continuous.*

(a) *If* $\mathbf{F}$ *is monotone on* $\mathcal{K}$*, the solution set of VI*$(\mathcal{K}, \mathbf{F})$ *is closed and convex (possibly empty).*

(b) *If* $\mathbf{F}$ *is strictly monotone on* $\mathcal{K}$*, the VI*$(\mathcal{K}, \mathbf{F})$ *has at most one solution.*

(c) *If* $\mathbf{F}$ *is strongly-monotone on* $\mathcal{K}$*, the VI*$(\mathcal{K}, \mathbf{F})$ *has a unique solution.*

(d) *If* $\mathbf{F}$ *is Lipschitz continuous and strongly-monotone on* $\Omega \supseteq \mathcal{K}$*, there exists a* $c > 0$ *such that* Ω

$$\|\mathbf{x} - \mathbf{x}^{\star}\| \leq c \left\|\mathbf{F}_{\mathcal{K}}^{nat}(\mathbf{x})\right\| \qquad \forall \mathbf{x} \in \Omega$$

where $\mathbf{x}^{\star}$ *is the unique solution of the VI and* $\mathbf{F}_{\mathcal{K}}^{nat}(\mathbf{x}) \triangleq \mathbf{x} - \prod_{\mathcal{K}} (\mathbf{x} - \mathbf{F}(\mathbf{x}))$ *(note that* $\mathbf{F}_{\mathcal{K}}^{nat}(\mathbf{x}^{\star}) = \mathbf{0}$*).*

- Remarks:
  - Strict monotonicity of $\mathbf{F}$ does not guarantee the existence of a solution. For example, $F(x) = e^x$ is strictly monotone, but the VI$(\mathbb{R}, e^x)$ does not have a solution.
  - Result in (d) provides an upper bound on the distance from the solution.
  - For "partitioned VI" (i.e., $\mathcal{K} = \prod_{i=1}^{Q} \mathcal{K}_i$, with $\mathcal{K}_i \subseteq \mathbb{R}^{n_i}$ and $\mathbf{F} = (\mathbf{F}_i)_{i=1}^{Q}$, with $\mathbf{F}_i : \mathcal{K}_i \to \mathbb{R}^{n_i}$), the existence and uniqueness results can be made weaker (there are necessary and sufficient conditions).

Parallelism with Convex Problems

Theorem. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex closed set, and let $f : \mathcal{K} \to \mathbb{R}$ be continuously differentiable ($\Rightarrow \mathbf{F} \triangleq \nabla f$ is continuous). Consider the optimization problem*

$$(P) : \qquad \underset{x \in \mathcal{K}}{\text{minimize}} \quad f(x)$$

(a) *If f is convex ($\Leftrightarrow \mathbf{F}$ is monotone) on $\mathcal{K}$, the solution set of (P) [the VI$(\mathcal{K}, \mathbf{F})$] is closed and convex.*

(b) *If f is strictly convex ($\Leftrightarrow \mathbf{F}$ is strictly monotone) on $\mathcal{K}$, problem (P) [the VI$(\mathcal{K}, \mathbf{F})$] has at most one solution.*

(c) *If f is strongly convex ($\Leftrightarrow \mathbf{F}$ is strongly-monotone) on $\mathcal{K}$, problem (P) [the VI$(\mathcal{K}, \mathbf{F})$] has a unique solution.*

Characterization of Solution of VI as a Projection

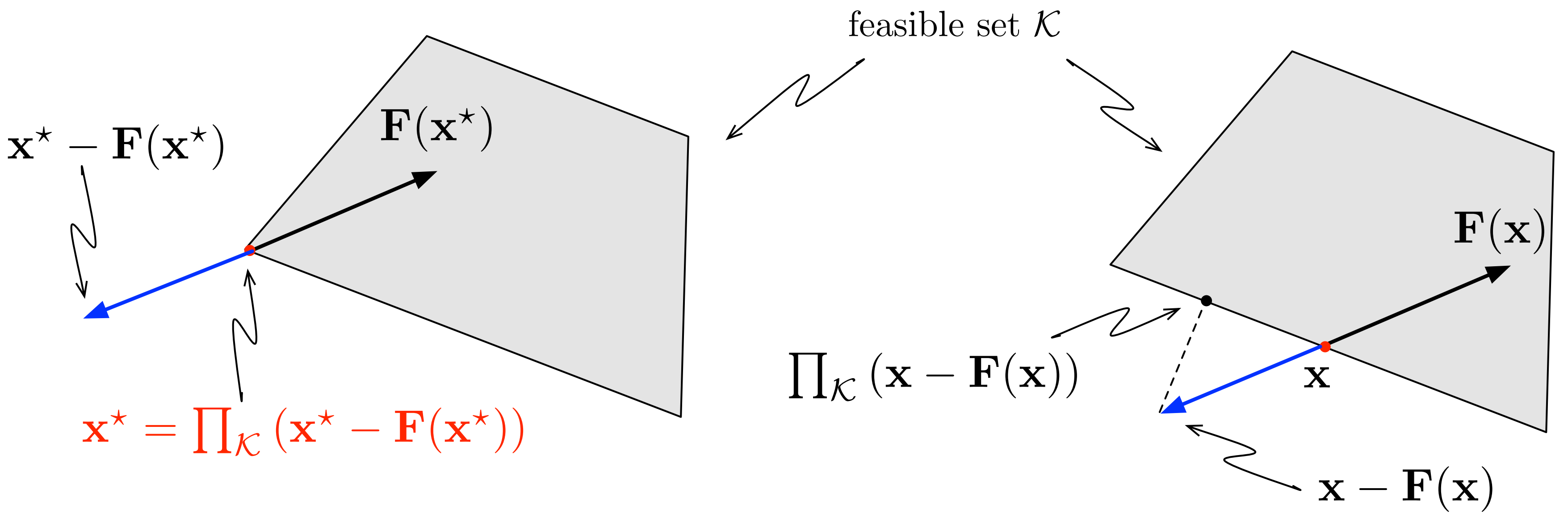
- The solution of a VI$(\mathcal{K}, \mathbf{F})$ can be interpreted as the Euclidean projection of a proper map onto the convex set $\mathcal{K}$.

- Let $\mathcal{K} \subseteq \mathbb{R}^n$ be closed and convex and let $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$ be arbitrary.

$$\mathbf{x}^\star \text{ is a solution of the VI}(\mathcal{K}, \mathbf{F}) \quad \Leftrightarrow \quad \mathbf{x}^\star = \prod\nolimits_{\mathcal{K}} (\mathbf{x}^\star - \mathbf{F}(\mathbf{x}^\star))$$

- The fixed-point interpretation can be very useful for both analytical and computational purposes:
  - Basic existence results of a solution of the VI coming from the fixed-point theorems (e.g., Corollary 2)
  - Development of a large family of iterative methods (e.g., projection methods).

- Graphical interpretation:
  - (a) $\mathbf{x}^\star$ is a solution of the VI$(\mathcal{K}, \mathbf{F})$ if and only if $\mathbf{x}^\star = \prod_{\mathcal{K}} (\mathbf{x}^\star - \mathbf{F}(\mathbf{x}^\star))$;
  - (b) a feasible $\mathbf{x}$ that is not a solution of the VI and thus $\mathbf{x} \neq \prod_{\mathcal{K}} (\mathbf{x}^\star - \mathbf{F}(\mathbf{x}^\star))$.



(a) (b)

Waterfilling: A New Look at an Old Result

- Capacity of parallel Gaussian channels $\{\lambda_k\}_{k=1}^N \geq 0$ under average power constraints:

$$\begin{array}{ll} \underset{\mathbf{p}}{\text{maximize}} & \sum_{k=1}^N \log\left(1 + p_k\,\lambda_k\right) \\ \text{subject to} & \sum_{k=1}^N p_k \leq P_T \\ & 0 \leq p_k \leq p_k^{\max}, \qquad 1 \leq k \leq N \end{array}$$

- Optimal power allocation: the waterfilling solution

$$p_k^{\star} = \left[\mu - \lambda_k^{-1}\right]_0^{p_k^{\max}}, \quad 1 \leq k \leq N$$

  where μ is found to satisfy $\sum_{k=1}^N p_k = P_T$.

- History: Shannon 1949, Holsinger 1964, Gallager 1968.

- For some problems, this waterfilling expression is not convenient.

New Interpretation of Waterfillings via VI

Theorem [ScuPhD'05, ScuPalBar'TSP07]**:** The waterfilling $\mathbf{p}^{\star} = \left[\mu - \boldsymbol{\lambda}^{-1}\right]_{\mathbf{0}}^{\mathbf{p}^{\max}}$ is the unique solution of the affine VI $(\mathcal{K}, \mathbf{F})$, where

$$\mathcal{K} = \left\{ \mathbf{p} \in \mathbb{R}^{N} : \sum_{i=1}^{N} p_k = P_T, \quad 0 \leq p_k \leq p_k^{\max} \text{ for all } k \right\},$$

and

$$\mathbf{F}(\mathbf{p}) = \mathbf{p} + \boldsymbol{\lambda}^{-1}.$$

Corollary [ScuPalBar'TSP08]**:** The waterfilling solution can be rewritten as a projection

$$\overbrace{\mathbf{p}^{\star} \in \text{SOL}(\mathcal{K}, \mathbf{F}) \Leftrightarrow \mathbf{p}^{\star} = \Pi_{\mathcal{K}}\left[\mathbf{p}^{\star} - \mathbf{F}\left(\mathbf{p}^{\star}\right)\right]}^{\mathbf{p}^{\star} = \Pi_{\mathcal{K}}\left[-\boldsymbol{\lambda}^{-1}\right]}$$

Algorithms for VI

Algorithms for VI

- Newton Methods for VIs
- Equation-Based Algorithms for Complementarity Problems
- KKT-Based Approaches for VIs
- Merit Function-Based Approaches for VIs
- Projection Methods for VIs
- Tikhonov Regularization and Proximal-Point Methods for VIs

Projection Methods: Introduction

- Projection methods are conceptually simple methods for solving monotone VI$(\mathcal{K}, \mathbf{F})$ for a convex closed set $\mathcal{K}$.

- Their advantages are
  - Easily implementable and computationally inexpensive (when $\mathcal{K}$ has structure that makes the projection onto $\mathcal{K}$ easy)
  - Suitable for large scale problems
  - They are often used as a sub-procedures in faster and more complex methods (enabling the moves into "promising" regions)

- Their main disadvantage is slow progress since they do not use higher order information.

Basic Idea of Projection Algorithms

- **Fact**: Recall that, given $\mathcal{K} \subseteq \mathbb{R}^n$ closed and convex, $\mathbf{x}^\star$ is a solution of the VI$(\mathcal{K}, \mathbf{F})$ if and only if $\mathbf{x}^\star$ is a fixed-point of the mapping $\mathbf{\Phi}(\mathbf{x}) \triangleq \prod_{\mathcal{K}}(\mathbf{x} - \mathbf{F}(\mathbf{x}))$, i.e., $\mathbf{x}^\star = \mathbf{\Phi}(\mathbf{x}^\star)$ [note that $\mathbf{\Phi} : \mathcal{K} \to \mathcal{K}$]

- The above fact motivates the following simple fixed-point based iterative scheme:

$$\mathbf{x}^{(n+1)} = \mathbf{\Phi}(\mathbf{x}^{(n)}), \qquad \mathbf{x}^{(0)} \in \mathcal{K}$$

  which produces a sequence with accumulation points being fixed points of $\mathbf{\Phi}$.

- If one could ensure that $\mathbf{\Phi}$ is a contraction in some norm, then one could use fixed-point iterates to find a fixed point of $\mathbf{\Phi}$ and hence, a solution to VI$(\mathcal{K}, \mathbf{F})$.

Basic Projection Method

Algorithm 1: Projection algorithm with constant step-size

`(S.0)` : Choose any $\mathbf{x}^{(0)} \in \mathcal{K}$, and the step size $\tau > 0$; set $n = 0$.

`(S.1)` : If $\mathbf{x}^{(n)} = \prod_{\mathcal{K}} \left(\mathbf{x}^{(n)} - \mathbf{F}(\mathbf{x}^{(n)})\right)$, then: `STOP`.

`(S.2)` : Compute

$$\mathbf{x}^{(n+1)} = \prod_{\mathcal{K}} \left(\mathbf{x}^{(n)} - \tau \, \mathbf{F}(\mathbf{x}^{(n)})\right).$$

`(S.3)` : Set $n \leftarrow n + 1$; go to `(S.1)`.

- In order to ensure the convergence of the sequence $\left\{\mathbf{x}^{(n+1)}\right\}_n$ (or a subsequence) to a fixed point of $\mathbf{\Phi}$, one needs some conditions of the mapping $\mathbf{F}$ and the step size τ. (Note that instead of a scalar step size, one can also use a positive definite matrix.)

Basic Projection Method: Convergence Conditions

- **Theorem.** *Let* $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$, *where* $\mathcal{K} \subseteq \mathbb{R}^n$ *is closed and convex. Suppose* $\mathbf{F}$ *is strongly monotone and Lipschitz continuous on* $\mathcal{K}$*:* $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$,

$$(\mathbf{x}-\mathbf{y})^T(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq c_{\mathbf{F}}\|\mathbf{x}-\mathbf{y}\|^2, \quad \text{and} \quad \|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})\| \leq L_{\mathbf{F}}\|\mathbf{x}-\mathbf{y}\|$$

and let

$$\tau < \frac{2c_{\mathbf{F}}}{L_{\mathbf{F}}^2}.$$

Then, the mapping $\prod_{\mathcal{K}}\left(\mathbf{x}^{(n)} - \tau\mathbf{F}(\mathbf{x}^{(n)})\right)$ *is a contraction in the Euclidean norm with contraction factor*

$$\eta = 1 - L_{\mathbf{F}}^2\,\tau\left(\frac{2c_{\mathbf{F}}}{L_{\mathbf{F}}^2} - \tau\right).$$

Therefore, any sequence $\left\{\mathbf{x}^{(n+1)}\right\}_n$ *generated by Algorithm 1 converges to the unique solution of the VI*$(\mathcal{K}, \mathbf{F})$.

- Pros:
  - Convergence of the *whole* sequence to the unique solution of the VI.
  - Easy to implement (for "nice" $\mathcal{K}$): one step of projection.

- Cons:
  - Strong requirements on $\mathbf{F}$ (strongly monotonicity and Lipschitz property).
  - We do not always have access to $L_{\mathbf{F}}$ and $c_{\mathbf{F}}$. Hence, we do not know how small τ should be to ensure convergence.

- We can trade the cons with a higher computational complexity; this includes using a variable stepsize, using the extra-gradient method, hyperplane projection methods, etc.

NEP as a VI

NEP as a VI: Solution Analysis and Algorithms

- We have seen that the NEP

$$\mathcal{G} =< \mathcal{K}, \mathbf{f} >: \qquad \begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i \end{array} \qquad \forall i = 1, \ldots, Q,$$

  is equivalent (under mild conditions) to the $\text{VI}(\mathcal{K}, \mathbf{F})$, where $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_Q$ and $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$.

- Building on the VI framework, we can now derive conditions for the existence/uniqueness of a NE and devise distributed algorithms along with their convergence properties.

- The state-of-the-art-results are given in [Scu-Facc-PanPal'IT10(sub)].

NEP as a VI: Existence and Uniqueness of a NE

Theorem. *Given the NEP* $\mathcal{G} = <\mathcal{K}, \mathbf{f}>$, *suppose that each* $\mathcal{K}_i$ *is closed and convex,* $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ *is continuously differentiable and convex in* $\mathbf{x}_i$, *for any* $\mathbf{x}_{-i}$, *and let* $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$. *Then the following statements hold.*

(a) *Suppose that for every* i *the strategy set* $\mathcal{K}_i$ *is bounded. Then the NEP [the* VI$(\mathcal{K}, \mathbf{F})$*] has a nonempty and compact solution set;*

(b) *Suppose that* $\mathbf{F}(\mathbf{x})$ *is a strictly monotone function on* $\mathcal{Q}$. *Then the NEP [the* VI$(\mathcal{K}, \mathbf{F})$*] has at most one solution;*

(c) *Suppose that* $\mathbf{F}(\mathbf{x})$ *is a strongly monotone on* $\mathcal{Q}$. *Then the NEP [the* VI$(\mathcal{K}, \mathbf{F})$*] has a unique solution.*

Matrix Conditions for the Monotonicity of $\mathbf{F}$

- We provide sufficient conditions for $\mathbf{F}$ to be (strictly/strongly) monotone.
- Let introduce the matrices $\mathsf{JF}_{\mathsf{low}}$ and $\mathbf{\Upsilon}_{\mathbf{F}}$, defined as

$$[\mathsf{JF}_{\mathsf{low}}]_{ij} \triangleq \inf_{\mathbf{x}\in\mathcal{K}} \begin{cases} \left|\nabla^2_{\mathbf{x}_i\mathbf{x}_i} f_i(\mathbf{x})\right|, & \text{if } i=j, \\ -\frac{1}{2}\left(\left|\nabla^2_{\mathbf{x}_i\mathbf{x}_j} f_i(\mathbf{x})\right| + \left|\nabla^2_{\mathbf{x}_j\mathbf{x}_i} f_j(\mathbf{x})\right|\right), & \text{otherwise.} \end{cases}$$

and

$$[\mathbf{\Upsilon}_{\mathbf{F}}]_{ij} \triangleq \begin{cases} \alpha_i^{\min}, & \text{if } i=j, \\ -\beta_{ij}^{\max}, & \text{otherwise,} \end{cases}$$

where

$$\alpha_i^{\min} \triangleq \inf_{\mathbf{z}\in\mathcal{K}} \lambda_{\mathsf{min}}\left(\nabla^2_{\mathbf{x}_i\mathbf{x}_i} f_i(\mathbf{x})\right) \quad \text{and} \quad \beta_{ij}^{\max} \triangleq \sup_{\mathbf{z}\in\mathcal{K}} \left\|\nabla^2_{\mathbf{x}_i\mathbf{x}_i} f_i(\mathbf{x})\right\|.$$

Proposition. [Scu-Facc-PanPal'IT10(sub)] *Let* $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$ *be continuously differentiable with bounded derivatives on* $\mathcal{K}$. *Then the following statements hold:*

(a) *If* $J\boldsymbol{F}_{low}$ *is a (strictly) copositive matrix or* $\boldsymbol{\Upsilon}_{\mathbf{F}}$ *is a positive semidefinite (definite) matrix, then* $\mathbf{F}$ *is* monotone *(*strictly monotone*) on* $\mathcal{K}$*;*

(b) *If* $J\boldsymbol{F}_{low}$ *or* $\boldsymbol{\Upsilon}_{\mathbf{F}}$ *is a positive definite matrix*, *then* $\mathbf{F}$ *is* strongly monotone *on* $\mathcal{K}$, *with strongly monotonicity constant* $c_{sm}(\mathbf{F}) \geq \lambda_{least}\left(J\boldsymbol{F}_{low}\right)$ [or $c_{\mathrm{sm}}(\mathbf{F}) \geq \lambda_{least}\left(\boldsymbol{\Upsilon}_{\boldsymbol{F}}\right)$].

- Sufficient conditions for $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$ are

$$\frac{1}{w_i}\sum_{j\neq i} w_j \frac{\beta_{ij}^{\max}}{\alpha_i^{\min}} < 1, \quad \forall i, \qquad \frac{1}{w_j}\sum_{i\neq j} w_i \frac{\beta_{ij}^{\max}}{\alpha_i^{\min}} < 1, \quad \forall j.$$

- Remarks:
  - The uniqueness result stated in part (b)-(c) of the Theorem does not require that the set $\mathcal{K}$ be bounded;
  - Note that if $\mathbf{\Upsilon_F} \succ \mathbf{0}$ (the NE is unique), it must be $\alpha_i^{\min} = \inf_{\mathbf{z}\in\mathcal{K}} \left[\lambda_{\min}(\nabla^2_{\mathbf{x}_i} f_i(\mathbf{z}))\right] > 0$ for all i. This implies the uniformly strong convexity of $f_i(\cdot, \mathbf{x}_{-i})$ for all $\mathbf{x}_{-i} \in \mathcal{K}_{-i}$ (the optimal solution of each player's optimizaition problem is unique);
  - The uniqueness conditions are sufficient also for global convergence of best-response asynchronous distributed algorithms described later on.

Algorithms for NEPs

- Algorithms for *strongly monotone* NEPs: Totally asynchronous best-response algorithm [Scu-Facc-PanPal'IT10(sub)].

- Algorithms for *monotone* NEPs: Proximal Decomposition Algorithms [Scu-Facc-PanPal'IT10(sub)].

Best-response Algorithms: Basic Idea

- A NE $\mathbf{x}^\star$ is defined as a "simultaneous" solution of each of the single-player optimization problems.

- Introducing the set of the optimal solutions to the i-th optimization problem for any given $\mathbf{x}_{-i}$

$$\mathcal{B}_i(\mathbf{x}_{-i}) \triangleq \{\mathbf{x}_i \in \mathcal{K}_i \,:\, f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \le f_i(\mathbf{y}_i, \mathbf{x}_{-i}), \quad \forall \mathbf{y}_i \in \mathcal{K}_i\}$$

  also termed as best-response map, the NEs of the NEP can be reinterpreted as the fixed-points of the best-response map $\mathcal{B}(\mathbf{x}) \triangleq (\mathcal{B}_i(\mathbf{x}_{-i}))_{i=1}^{Q}$, i.e.,

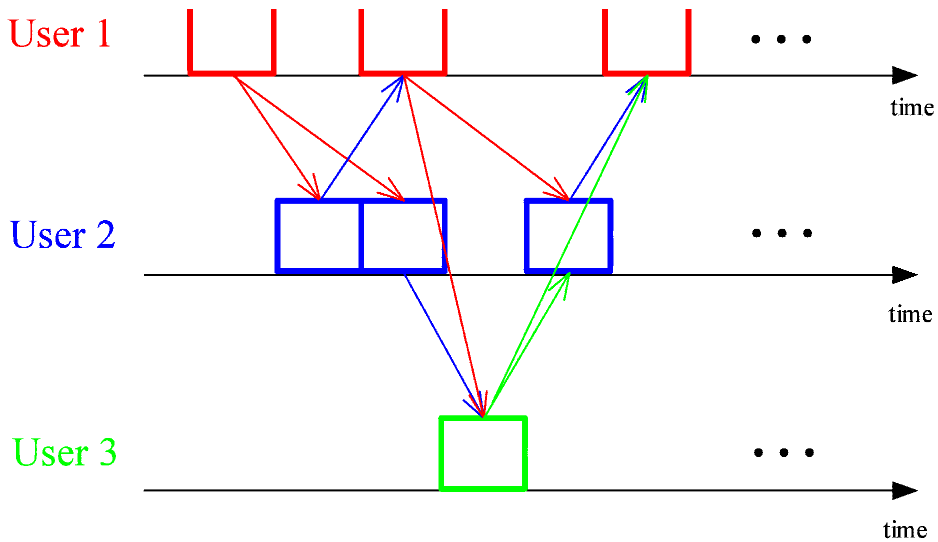
$$\mathbf{x}^\star \in \mathcal{B}(\mathbf{x}^\star).$$

- This fact motivates the class of so-called "best-response algorithms": fixed-point based iterative schemes where at every iteration each player updates his own strategy choosing a point in $\mathcal{B}_i(\mathbf{x}_{-i})$ (solving his own optimization problem given $\mathbf{x}_{-i}$).

- If one could ensure that $\mathcal{B}$ is a contraction in some norm, then one could use fixed-point iterates to find a fixed point of $\mathcal{B}$ and hence, a solution to the NEP.

Asynchronous Best-Response Algorithm

- In the Asynchronous Best-Response Algorithm, users
  - update their strategy in a totally asynchronous way based on $\mathcal{B}_i(\mathbf{x}_{-i})$;
  - may update at arbitrary and different times and more or less frequently than others
  - may use an outdated measure of the strategies of the others.



Asynchronous Best-Response Algorithm

Algorithm 2: Asynchronous Best-Response Algorithm

(S.0) : Choose any feasible starting point $\mathbf{x}^{(0)} = (\mathbf{x}_i^{(0)})_{i=1}^{Q}$; set $n = 0$.

(S.1) : If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: STOP

(S.2) : for $i = 1, \ldots, Q$, compute $\mathbf{x}_i^{(n+1)}$ as

$$\mathbf{x}_i^{(n+1)} = \begin{cases} \mathbf{x}_i^{\star} \in \underset{\mathbf{x}_i \in \mathcal{K}_i}{\operatorname{argmin}}\, f_i\left(\mathbf{x}_i,\, \mathbf{x}_{-i}^{(\tau^i(n))}\right), & \text{if } n \in \mathcal{T}_i, \\ \mathbf{x}_i^{(n)}, & \text{otherwise} \end{cases}$$

end

(S.3) : $n \leftarrow n+1$; go to (S.1).

Theorem. [Scu-Facc-PanPal'IT10(sub)] *Given the NEP* $\mathcal{G} = <\mathcal{K}, \mathbf{f}>$*, suppose that* $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$. Then, *any sequence* $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ *generated by the asynchronous best-response algorithm converges to the unique NE of* $\mathcal{G}$*, for any given updating feasible schedule of the players.*

- Remarks:
  - Under $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$, the best-response mapping $\mathcal{B}(\mathbf{x})$ is a contraction (or equivalently, the function $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$ is strongly monotone on $\mathcal{K}$);
  - If $f_i(\cdot, \mathbf{x}_{-i})$ is not uniformly strongly convex for all $\mathbf{x}_{-i} \in \mathcal{K}_{-i}$, (i.e., $\boldsymbol{\Upsilon}_{\mathbf{F}} \not\succ \mathbf{0}$), the algorithm is not guaranteed to converge.

- How to deal with (non strongly) monotone NEPs?

Algorithms for Monotone NEPs: Main Idea

- We know how to solve a strongly monotone NEP. To solve a monotone NEP the proposed approach is then to make it strongly monotone by a proper regularization.

- The regularization has to be chosen so that:
  - one can recover somehow the NEs of the original game from those of the regularized game;
  - one should be able to solve a regularized game via distributed algorithms.

- Ingredients:
  - Equivalence between the VI and the NEP;
  - Proximal decomposition algorithms for monotone VIs.

GNEP as a VI

GNEP: Basic Definitions

- The GNEP extends the classical NEP setting by assuming that each player's strategy set can depend on the rival players' strategies $\mathbf{x}_{-i}$.

- Let $\mathcal{K}_i(\mathbf{x}_{-i}) \subseteq \mathbb{R}^{n_i}$ be the feasible set of player i when the other players choose $\mathbf{x}_{-i}$. The aim of each player i, given $\mathbf{x}_{-i}$, is to choose a strategy $\mathbf{x}_i \in \mathcal{K}_i(\mathbf{x}_{-i})$ that solves the problem

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i(\mathbf{x}_{-i}). \end{array}$$

- A Generalized Nash Equilibrium (GNE) is a feasible point $\mathbf{x}^\star$ such that the following holds for each player $i = 1, \ldots, Q$:

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathcal{K}_i(\mathbf{x}_{-i}^\star).$$

- Facts:
  - The GNEP can be rewritten as a QVI.
  - However, QVIs are much harder problems than VIs and only few results are available;
  - Thus the GNEP, in its full generality, is almost intractable and also the VI approach does not help.

- We then restrict our attention to particular classes of (more tractable) equilibrium problems: the so-called GNEPs with *jointly convex shared constraints.*

GNEPs With Jointly Convex Shared Constraints (JCSC)

- A GNEP is termed as *GNEP with jointly convex shared constraints* if the feasible sets are defined as:

$$\mathcal{K}_i(\mathbf{x}_{-i}) \triangleq \left\{\mathbf{x}_i \in \overline{\mathcal{K}}_i \,:\, \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}\right\}$$

  where:

  – $\overline{\mathcal{K}}_i \subseteq \mathbb{R}^{n_i}$ is the closed and convex set of individual constraints of player i;
  – $\mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}$ represents the set of shared coupling constraints (*equal for all the players*);
  – $\mathbf{g} \triangleq (g_j)_{j=1}^m : \mathbb{R}^n \to \mathbb{R}^m$ is (jointly) convex in $\mathbf{x}$.

- If there are no shared constraints the GNEP reduces to a NEP.

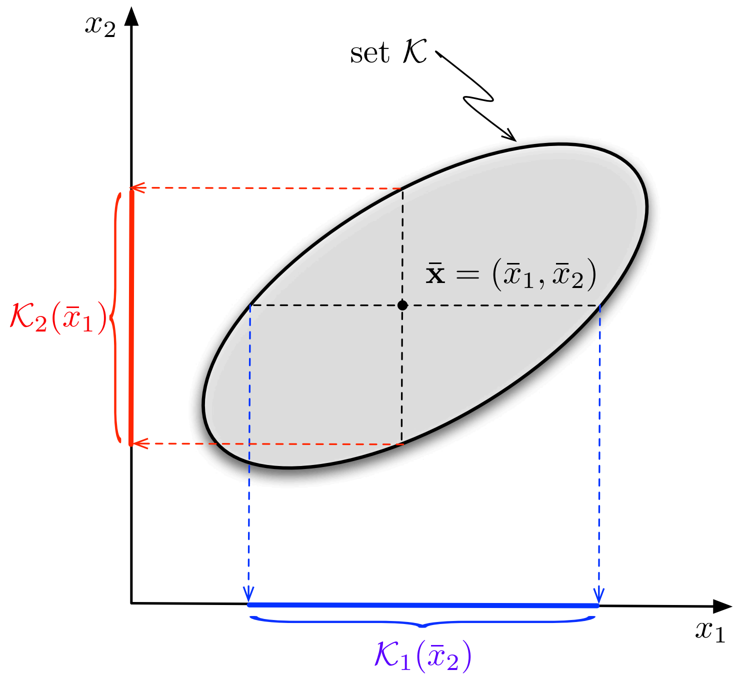
Geometrical Interpretation

- Let define the joint feasible set $\mathcal{K}$

$$\mathcal{K} = \left\{ \mathbf{x} \,:\, \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}, \quad \mathbf{x}_i \in \overline{\mathcal{K}}_i, \, \forall i = 1, \ldots, Q \right\}$$

- It easy to see that

$$\mathcal{K}_i(\mathbf{x}_{-i}) = \left\{ \mathbf{x}_i \,:\, (\mathbf{x}_i, \, \mathbf{x}_{-i}) \in \mathcal{K} \right\}.$$



Connection Between GNEPs with JCSC and VIs

- GNEPS with JCSC are still very difficult problems, however at least some type of solutions can be studied and calculated using the VI approach.

- Given the GNEP with JCSC, let

$$\mathcal{K} = \left\{ \mathbf{x} \,:\, \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}, \quad \mathbf{x}_i \in \overline{\mathcal{K}}_i, \, \forall i \right\}, \quad \text{and} \quad \mathbf{F} = \left(\nabla_{\mathbf{x}_i} f_i\right)_{i=1}^{Q}$$

  and consider the VI$(\mathcal{K}, \mathbf{F})$.

- **Lemma.** *Every solution of the VI$(\mathcal{K}, \mathbf{F})$ is a solution of the GNEP with JCSC; the converse in general may not be true.*

- The solutions of the GNEP that are also solution of the VI are termed as *Variational Equilibria* (VE).

VEs: Game Theoretical Interpretation

- Recall that the VEs are solution of the VI, with

$$\mathcal{K} = \left\{\mathbf{x} : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \mathbf{0}, \quad \mathbf{x}_i \in \overline{\mathcal{K}}_i, \forall i\right\}, \quad \text{and} \quad \mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}.$$

- Connection between the VI$(\mathcal{K}, \mathbf{F})$ and NEPs: $\mathbf{x}^\star \in \mathcal{K}$ is a VE if and only if $\mathbf{x}^\star$ along with a suitable $\boldsymbol{\lambda}^\star$is a solution of the NEP

$$\mathcal{G}_{\boldsymbol{\lambda}^\star} : \qquad \begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) + \boldsymbol{\lambda}^{\star T}\mathbf{g}(\mathbf{x}) \\ \text{subject to} & \mathbf{x}_i \in \overline{\mathcal{K}}_i \end{array} \qquad \forall i = 1, \ldots, Q,$$

and furthermore

$$\mathbf{0} \leq \boldsymbol{\lambda}^\star \perp \mathbf{g}(\mathbf{x}^\star) \leq \mathbf{0}.$$

- Remarks:
  - We can interpret the $\boldsymbol{\lambda}^{\star}$ as prices paid by the players for the common "resource" represented by the shared constraints;
  - The complementarity condition says that we actually have to pay only when the resource becomes scarce;
  - Thus, the NEP with pricing can be seen as a "penalized" version of the GNEP with JCSC, where the shared constraints are enforced by making the players to pay the *common* price $\boldsymbol{\lambda}^{\star}$;
  - Mathematically, $\boldsymbol{\lambda}^{\star}$ is the KKT common multiplier of the shared constraints.

- We are now able to reduce the solution analysis & computation of a VE to that of the equilibrium of a NEP, to which we can in principle apply the theory developed so far.

VEs: Solution Analysis and Algorithms

- **Solution analysis**: Since the VEs are solution of a VI, one can derive existence and uniqueness conditions from the VI theory developed so far.

- **Algorithms**: Similarly, we can also devise algorithms for VEs based on the VI$(\mathcal{K}, \mathbf{F})$; however they will not be distributed since the set $\mathcal{K}$ does not have a Cartesian structure (there is a coupling among the strategies of the players).

- **How to attack the problem**: Building on the equivalence between the VI$(\mathcal{K}, \mathbf{F})$ and the NEP with pricing, we can overcome this issue. To do that, however, we still need some more work.

Toward Distributed Algs: A NCP Reformulation

- We rewrite the NEP with pricing as a VI whose feasible set has a Cartesian structure. For the sake of simplicity, we focus only on strongly monotone games $[\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}]$.

- **Step 1**: The NEP with pricing can be rewritten as

$$\mathcal{G}_{\boldsymbol{\lambda}}: \quad \begin{array}{l} \text{VI}(\overline{\mathcal{K}}, \mathbf{F} + \nabla \mathbf{g}^T \boldsymbol{\lambda}) \\ \mathbf{0} \leq \boldsymbol{\lambda} \perp \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \end{array}$$

  where $\overline{\mathcal{K}} = \prod_{i=1}^{Q} \overline{\mathcal{K}}_i$ and $\mathbf{F} = (\nabla_{\mathbf{x}_i} f_i)_{i=1}^{Q}$.

- The $\text{VI}(\overline{\mathcal{K}}, \mathbf{F} + \nabla \mathbf{g}^T \boldsymbol{\lambda})$ is strongly monotone and thus has a unique solution $\mathbf{x}^\star(\boldsymbol{\lambda})$ [the unique NE of $\mathcal{G}_{\boldsymbol{\lambda}}$].

- **Step 2**: We rewrite $\mathcal{G}_{\boldsymbol{\lambda}} \cup \text{CC}$ as a NCP.
  Let define the map

$$\boldsymbol{\Phi}(\boldsymbol{\lambda}) : \mathbb{R}_{+}^{m} \ni \boldsymbol{\lambda} \rightarrow -\mathbf{g}\left(\mathbf{x}^{\star}(\boldsymbol{\lambda})\right)$$

which measures the violation of the shared constraints at $\mathbf{x}^{\star}(\boldsymbol{\lambda})$.

- **Theorem. [Scu-Facc-PanPal'IT10(sub)]** *If* $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$, *the (strongly monotone) NEP with pricing in* $\mathcal{G}_{\boldsymbol{\lambda}} \cup \text{CC}$ *is equivalent to the NCP in the price tuple* $\boldsymbol{\lambda}$

$$\text{NCP}(\boldsymbol{\Phi}) : \qquad \mathbf{0} \leq \boldsymbol{\lambda} \perp \boldsymbol{\Phi}(\boldsymbol{\lambda}) \geq \mathbf{0} \quad \Leftrightarrow \quad \textit{VI}(\mathbb{R}_{+}^{m}, \boldsymbol{\Phi}).$$

- The NCP reformulation is instrumental to devise distributed algorithms. We can now use the algorithms developed so far for strongly monotone VIs.

Distributed Projection Algorithms based on NCP

Algorithm 3: Projection Algorithm with Variable Steps (PAVS)

(S.0) : Choose any $\boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$; set $n = 0$.

(S.1) : If $\boldsymbol{\lambda}^{(n)}$ satisfies a suitable termination criterion: STOP.

(S.2) : Given $\boldsymbol{\lambda}^{(n)}$, compute $\mathbf{x}^{\star}(\boldsymbol{\lambda}^{(n)})$ as the unique NE of $\mathcal{G}_{\boldsymbol{\lambda}^{(n)}}$:

$$\mathbf{x}^{\star}(\boldsymbol{\lambda}^{(n)}) = \text{SOL}(\overline{\mathcal{K}}; \mathbf{F} + \nabla_{\mathbf{x}}\mathbf{g}\boldsymbol{\lambda}^{(n)}).$$

(S.3) : Choose $\tau_n > 0$ and update the price vectors $\boldsymbol{\lambda}$ according to

$$\boldsymbol{\lambda}^{(n+1)} = \left[\boldsymbol{\lambda}^{(n)} - \tau_n \, \boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)\right]^{+}.$$

(S.4) : Set $n \leftarrow n + 1$; go to (S.1).

- **Theorem. [Scu-Facc-PanPal'IT10(sub)]** *Suppose* $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$. *If the scalars* τ_n *are chosen so that* $0 < \inf_n \tau_n \leq \sup_n \tau_n < 2\, c_{coc}(\boldsymbol{\Phi})$, *where* $c_{coc}(\boldsymbol{\Phi}) \triangleq \hat{c}_{sm}(\mathbf{F})/c_{Lip}^2(\mathbf{g})$, $c_{Lip}(\mathbf{g}) \triangleq \max_{\mathbf{x}\in\overline{\mathcal{K}}} \left\|\nabla \mathbf{g}(\mathbf{x})^T\right\|_2$, *and* $\hat{c}_{sm}(\mathbf{F})$ *is the strongly-monotonicity constant of* $\mathbf{F}$, *then the sequence* $\{\boldsymbol{\lambda}^{(n)}\}_{n=0}^{\infty}$ *generated by the PAVS converges to a solution of the NCP*($\boldsymbol{\Phi}$).

- **Inner loop**: The NE $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ of $\mathcal{G}_{\boldsymbol{\lambda}^{(n)}}$ can be computed using the asynchronous best-response algorithms (convergence is guaranteed under $\boldsymbol{\Upsilon}_{\mathbf{F}} \succ \mathbf{0}$).

- **Algorithms for** $\boldsymbol{Monotone}$ **Games:** Following the same idea as for monotone NEPs we can make the monotone NEP with pricing strongly-monotone via Proximal regularization [Scu-Facc-PanPal'IT10(sub)].

Part III:

Variational Inequality Theory: Applications

Part III - Outline

- Application of NEP: Ad-Hoc Networks
- Application of GNEP: QoS Ad-Hoc Networks
- Application of NEP: Robust CR Systems with Individual Constraints
- Application of NEP: Robust CR Systems with Individual Constraints
- Application of GNEP with Shared Constraints: Cognitive Radio Systems
- Application of GNEP with Shared Constraints: Routing in Communication Networks

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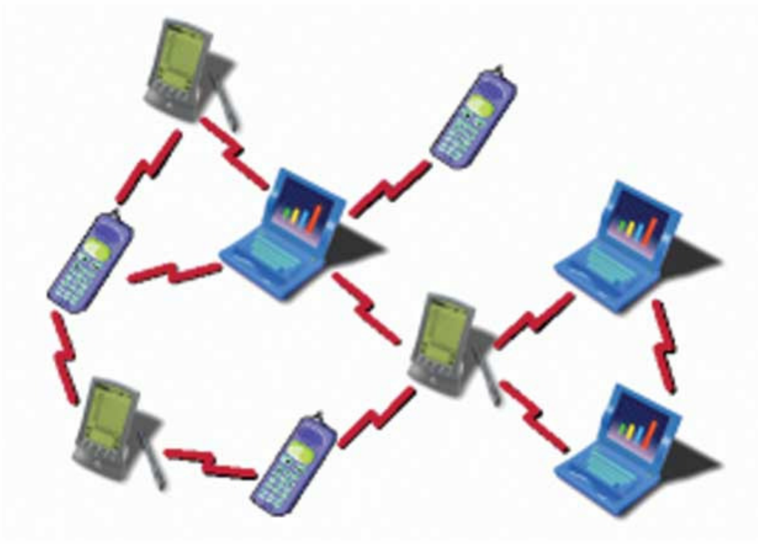
Application of NEP: Ad-Hoc Networks

Competitive Ad-Hoc Networks

- Consider a decentralized and competitive network of users fighting for the resources (i.e., spectrum):



- Baseband signal model: Vector Gaussian Interference Channel (IC)

$$\mathbf{y}_q = \underbrace{\mathbf{H}_{qq}\mathbf{x}_q}_{\text{desired signal}} + \underbrace{\sum_{r \neq q} \mathbf{H}_{qr}\mathbf{x}_r}_{\text{inteference}} + \mathbf{w}_q.$$

The Frequency-Selective Case

- The channel matrices are diagonal: $\mathbf{H}_{qr} = \text{diag}\left\{(H_{qr}(k))_{k=1}^{N}\right\}$.
- The optimization variables correspond to the power allocation over the carriers: $\mathbf{p}_q = \{p_q(k)\}_{k=1}^{N}$.
- There is a power budget for each user:

$$\sum_{k=1}^{N} p_q(k) \leq P_q.$$

- The payoff for user q is the information rate:

$$r_q\left(\mathbf{p}_q, \mathbf{p}_{-q}\right) = \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_q(k)\right) \quad \text{with} \quad \mathsf{sinr}_q(k) = \frac{\left|H_{qq}(k)\right|^2 p_q(k)}{1 + \sum_{r \neq q} \left|H_{qr}(k)\right|^2 p_r(k)}.$$

- The feasible set for the variables is: $\mathcal{P}_q = \left\{\mathbf{p}_q \in \mathbb{R}_+^N : \sum_{k=1}^{N} p_q(k) = P_q\right\}$.

Game Formulation for Ad-Hoc Networks

- Each of the Q users selfishly maximizes its own rate subject to the constraints:

$$\mathcal{G}: \quad \begin{array}{ll} \underset{\mathbf{p}_q}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1 + \text{sinr}_q(k)\right) \\ \text{subject to} & \mathbf{p}_q \in \mathcal{P}_q \end{array} \qquad q = 1, \ldots, Q$$

- Given the strategies of the others $\mathbf{p}_{-q}$, the best response for each user is the waterfilling solution:

$$\mathbf{p}_q^{\star} = \text{wf}_q\left(\mathbf{p}_{-q}\right) \triangleq \left(\mu_q - \textbf{interf}_q(\mathbf{p}_{-q})\right)^{+}$$

where

$$\text{interf}_q(k; \mathbf{p}_{-\mathbf{q}}) \triangleq \frac{1 + \sum_{r \neq q} \left|H_{qr}(k)\right|^2 p_r(k)}{\left|H_{qq}(k)\right|^2} \qquad k = 1, \ldots, N$$

- Any NE is a simultaneous waterfilling for all users:

$$\mathbf{p}_q^\star = \mathsf{wf}_q\left(\mathbf{p}_{-q}^\star\right) \qquad \forall q = 1, \ldots, Q \quad \Longleftrightarrow \quad \mathbf{p}^\star = \mathsf{wf}\left(\mathbf{p}^\star\right).$$

- Main issues:
  - Does a NE exist?
  - Is the NE unique?
  - How to reach a NE?

- Open problem for years: Different researchers have actively worked on this problem since 2001 [Yu-Gin-Ciof'01], [Yam-Luo'04], [Luo-Pang'06], [Scu-Pal-Bar'06], ...

- VI provides the answers.

How to Attack This Problem?

- One option is to write the NEP using the minimum principle as a VI and try to show monotonicity properties for the function $\mathbf{F}$.

- A more elaborate option is to write the KKT conditions of the NEP and, after some manipulations, rewrite the NEP as an Affine VI and try to show monotonicity properties.

- In this problem, since the best-response of the NEP can be written in closed-form, we could try to show a contraction mapping property.

- Interestingly, the fixed-point characterization of the solution of the AVI happens to be the alternative simple representation of the waterfilling solution as a projection.

- This results in showing some norm property on a matrix that is equivalent to showing strongly monotonicity of the AVI.

- **Why is so difficult studying this game?**
  - We need to prove that the waterfilling mapping is a contraction:

$$\left\|\mathsf{wf}\left(\mathbf{p}^{(1)}\right)-\mathsf{wf}\left(\mathbf{p}^{(2)}\right)\right\| \overset{?}{\leq} c\left\|\mathbf{p}^{(1)}-\mathbf{p}^{(2)}\right\|$$

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$$\left\| \mathsf{wf}\left(\mathbf{p}^{(1)}\right) - \mathsf{wf}\left(\mathbf{p}^{(2)}\right) \right\| \overset{?}{\leq} c \left\| \mathbf{p}^{(1)} - \mathbf{p}^{(2)} \right\|$$

  - Using the definition of the wf:

$$\begin{aligned} \left\| \mathsf{wf}\left(\mathbf{p}^{(1)}\right) - \mathsf{wf}\left(\mathbf{p}^{(2)}\right) \right\| &= \left\| \left(\boldsymbol{\mu}^{(1)} - \mathbf{M}\mathbf{p}^{(1)}\right)^{+} - \left(\boldsymbol{\mu}^{(2)} - \mathbf{M}\mathbf{p}^{(2)}\right)^{+} \right\| \\ &\leq \left\| \left(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}\right) - \left(\mathbf{M}\mathbf{p}^{(1)} - \mathbf{M}\mathbf{p}^{(2)}\right) \right\| \end{aligned}$$

We are stuck !!!

- **Why is so difficult studying this game?**
  - We need to prove that the waterfilling mapping is a contraction:

$$\left\| \mathsf{wf}\left(\mathbf{p}^{(1)}\right) - \mathsf{wf}\left(\mathbf{p}^{(2)}\right)\right\| \overset{?}{\leq} c\left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|$$

  - Using our interpretation of the wf as a projector:

$$\begin{aligned}\left\| \mathsf{wf}\left(\mathbf{p}^{(1)}\right) - \mathsf{wf}\left(\mathbf{p}^{(2)}\right)\right\| &= \left\|\Pi_{\mathcal{K}}\left(-\mathbf{M}\mathbf{p}^{(1)}\right) - \Pi_{\mathcal{K}}\left(-\mathbf{M}\mathbf{p}^{(2)}\right)\right\| \\ &\leq \left\|\mathbf{M}\left(\mathbf{p}^{(2)} - \mathbf{p}^{(1)}\right)\right\| \\ &\leq \|\mathbf{M}\| \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|\end{aligned}$$

We are done !!!

Existence and Uniqueness of the NE

- **Theorem [Scu-Pal-Bar'TSP08]:** The solution set of the game is nonempty and compact. The NE is unique if

$$\rho\left(\mathbf{H}^{\max}\right) < 1$$

where

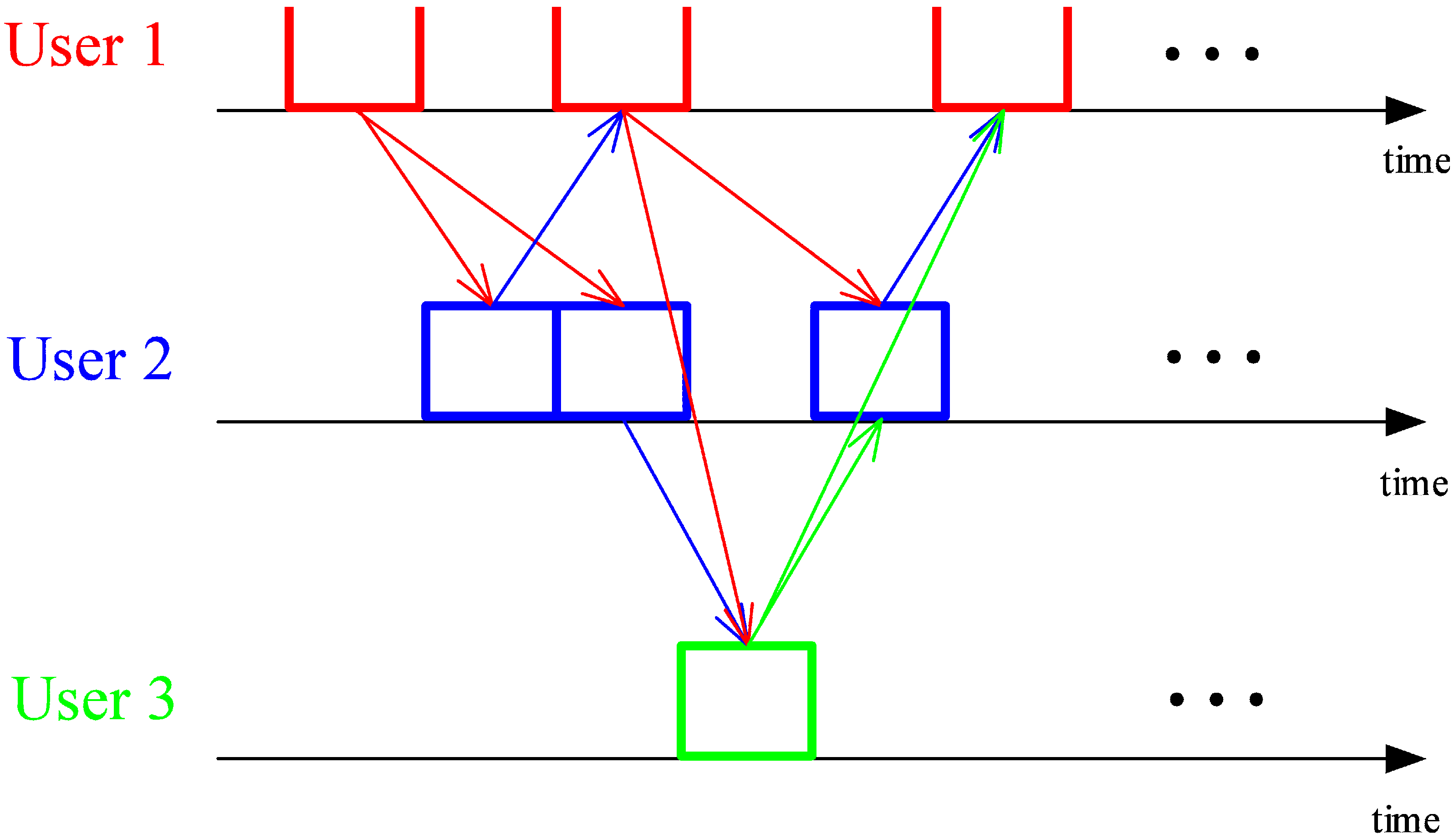
$$\left[\mathbf{H}^{\max}\right]_{qr} \triangleq \begin{cases} \max_k \left\{ \dfrac{|H_{qr}(k)|^2}{|H_{qq}(k)|^2} \right\} & \text{if } q \neq r \\ 0 & \text{otherwise.} \end{cases}$$

- Sufficient conditions:

Low MUI : $$\sum_{r \neq q} \max_k \left\{ \frac{|H_{qr}(k)|^2}{|H_{qq}(k)|^2} \right\} < 1, \quad \forall q = 1, \ldots, Q.$$

State-of-the-Art Algorithms: Asynchronous IWFA

- Do the players reach an equilibrium if every one selfishly performs his waterfilling solution $\mathsf{wf}_q\left(\mathbf{p}_{-q}\right)$ against the others?



State-of-the-Art Algorithms: Asynchronous IWFA

- Do the players reach an equilibrium if every one selfishly performs his waterfilling solution $\mathrm{wf}_q(\mathbf{p}_{-q})$ against the others?

- **Asynchronous IWFA [Scu-Pal-Bar'IT08]**: Users update the power allocation in a totally asynchronous way based on $\mathrm{wf}_q(\cdot)$
  - users may update at arbitrary and different times and more or less frequently than others
  - users may use an outdated measure of interference
  - distributed implementation: local measures of the MUI & weak constraints on synchronization

- **Theorem [Scu-Pal-Bar'IT08]:** The asynchronous IWFA converges to the unique NE if $\rho(\mathbf{H}^{\max}) < 1$.

The MIMO Case

Ad-Hoc Networks - The MIMO Game

- Game theoretic formulation [Scu-Pal-Bar'TSP08]:

$$\begin{array}{lll} \underset{\mathbf{Q}_q}{\text{maximize}} & \log\det\left(\mathbf{I}+\mathbf{Q}_q\mathbf{H}_{qq}^{\dagger}\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})\mathbf{H}_{qq}\right) & \\ \text{subject to} & \mathbf{Q}_q \succeq 0, \ \ \text{Tr}\left(\mathbf{Q}_q\right) \leq P_q & \end{array} \qquad q=1,\ldots,Q$$

where $\mathbf{R}_{-q}(\mathbf{Q}_{-q}) = \mathbf{R}_{n_q} + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{Q}_r\mathbf{H}_{rq}^{\dagger}$.

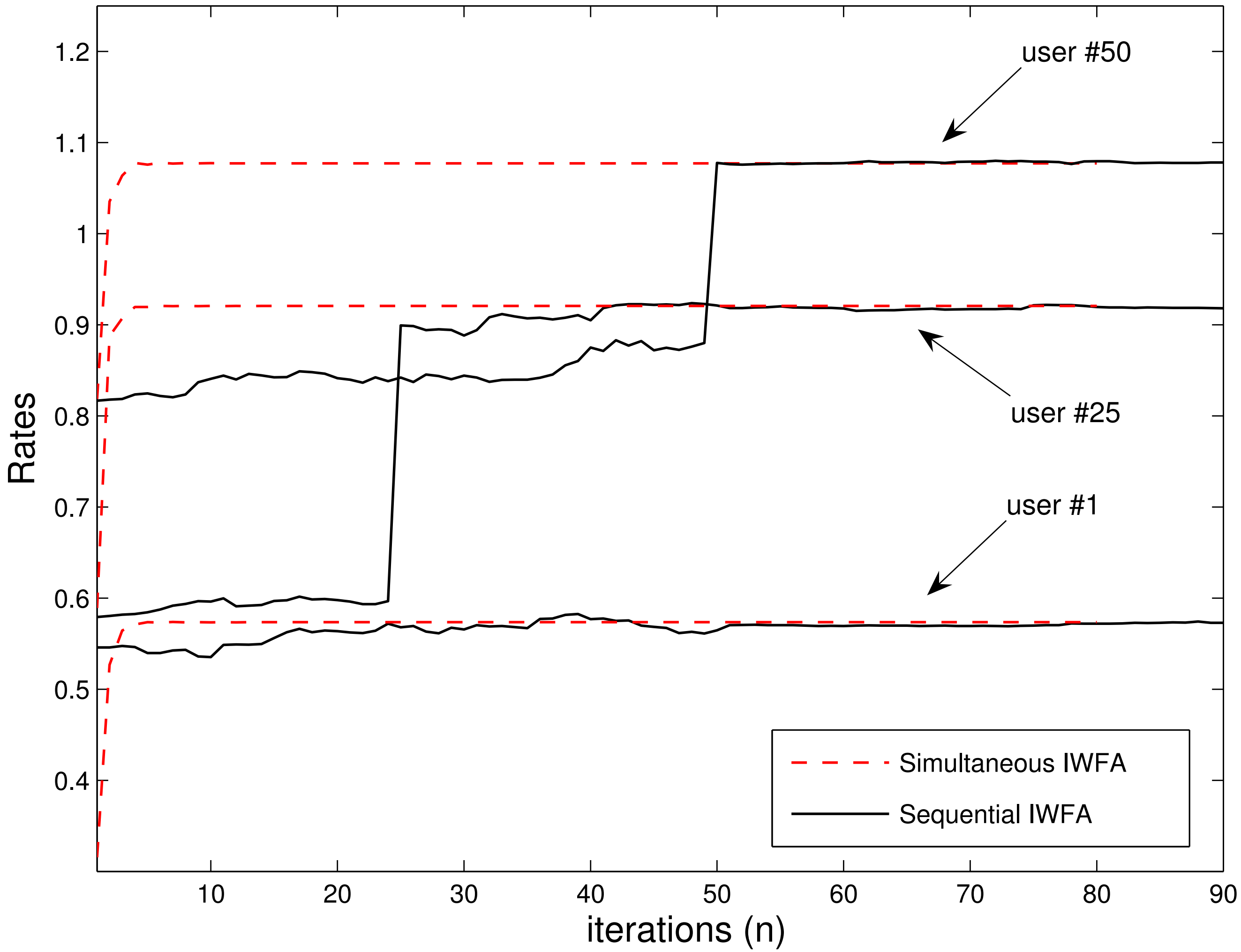
- Can we similarly analize this MIMO formulation?

- The answer is affirmative when the channel matrices are square and invertible.

Difficulties in the MIMO Case

- However, the answer is negative for the general case!!
- Some expected conjectures do not hold when the channel matrices are not square:
  - the WF as a projection does not follow from the square case simply replacing the inverse with some generalized inverse
  - the VI cannot be rewritten as a AVI and the reverse order law for generalized inverses does not hold
  - the multiuser WF is not a contraction under the conditions valid for the square case.
- Full characterization of the game for *arbitrary* MIMO channels (not square, not invertible) [Scu-Pal-Bar'TSP09]:
  - Solution analysis: Existence and uniqueness of the NE
  - Distributed algorithms: Asynchronous MIMO IWFA

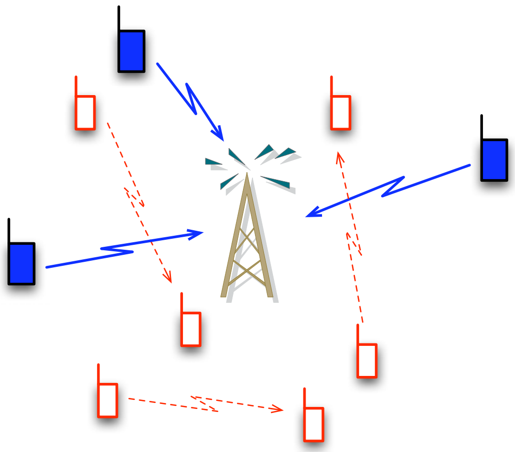
Convergence of the IWFA



Application of NEP: CR Systems with Individual Constraints

CR Systems

- Consider now an established network of primary users on top of which some secondary users play the previous game.

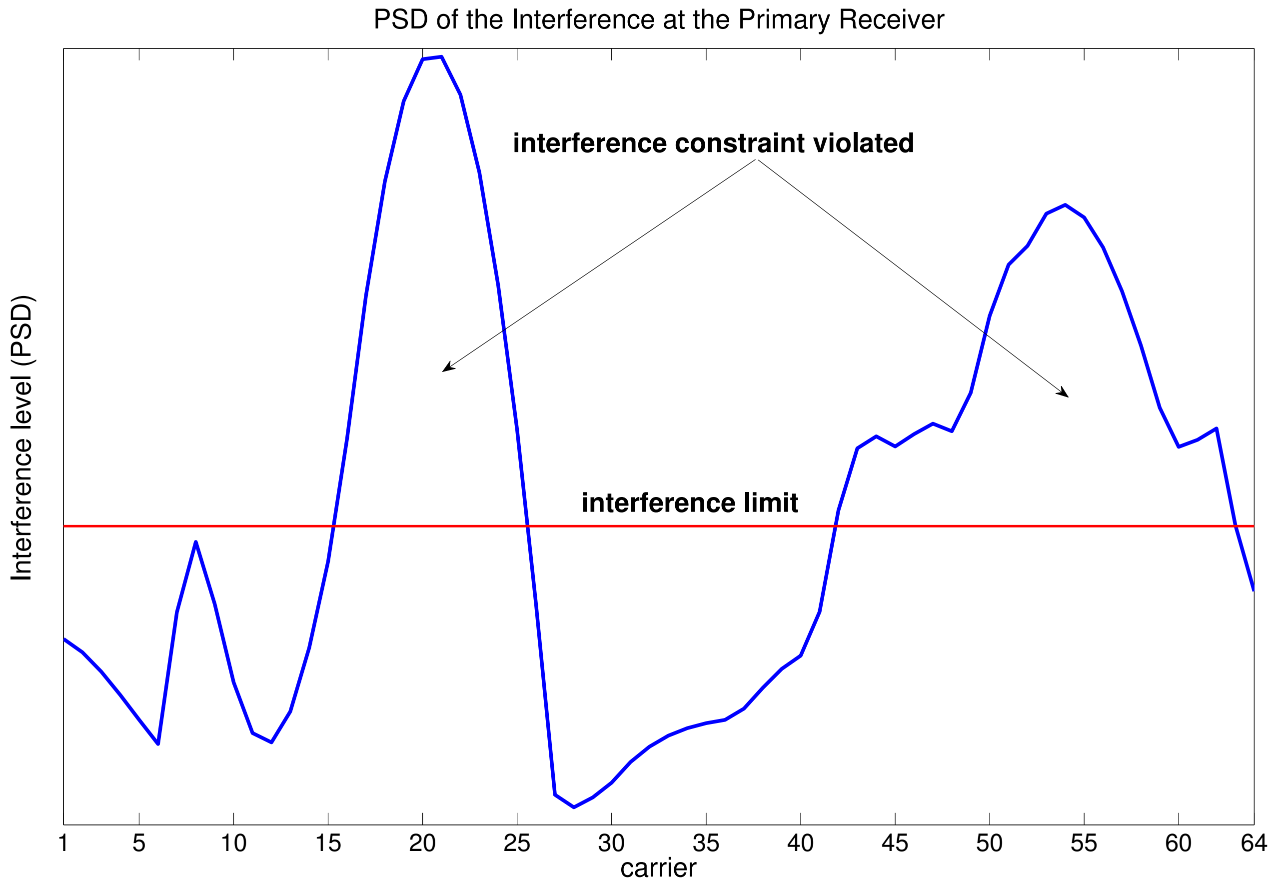


- Hierarchical CR networks
  - PU=Primary users (legacy spectrum holders)
  - SU=Secondary users (unlicensed users).

- Opportunistic communications: SUs can use the spectrum subject to inducing a limited interference or no interference at all on PUs.

Standard Waterfilling Does Not Work

- The standard iterative waterfilling algorithm does not work because it violates the interference constraint:



Signal Model for CR Systems

- Same signal model as for ad-hoc networks with the additional *individual* per-carrier interference constraints:

$$\left|h_{qp}^{(P,S)}(k)\right|^2 p_q(k) \leq I_{pq}(k), \quad k = 1, \ldots, N \quad p = 1, \ldots, P$$

  - $|h_{qp}^{(P,S)}(k)|^2$ is the cross-channel gain between the qth secondary and the pth primary user
  - $I_{pq}(k)$ is the maximum level of interference tolerable by the primary user p from the secondary user q over the subchannel k.

- Equivalently, we can write these constraints in terms of mask constraints

$$p_q(k) \leq p_q^{\max}(k) \triangleq \min_{p=1,\ldots,P} \frac{I_{pq}(k)}{\left|h_{qp}^{(P,S)}(k)\right|^2} \quad k = 1, \ldots, N.$$

Game Formulation for CR Systems

- Each of the Q users selfishly maximizes its own rate subject to the constraints:

$$\begin{array}{lll} \underset{\mathbf{p}_q}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_q(k)\right) & \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q & \forall q = 1, \ldots, Q \\ & 0 \leq p_q(k) \leq p_q^{\max}(k), \quad \forall k & \end{array}$$

- The best response in this case also has a nice and simple closed-form expression based on a modified waterfilling with clipping from above:

$$\widetilde{\mathsf{wf}}_q\left(\mathbf{p}_{-q}\right) \triangleq \left[\mu_q - \textbf{interf}_q(\mathbf{p}_{-q})\right]_{\mathbf{0}}^{\mathbf{p}^{\max}}.$$

- Similar analysis and algorithms as for the previous game, based on our interpretation of $\widetilde{\mathsf{wf}}$ as a projection [Scu-Pal-Bar'TSP07].

Application of GNEP with Shared Constraints: Cognitive Radio Systems

Signal Model for CR Systems

- Same signal model as for ad-hoc networks with the additional *individual* per-carrier interference constraints:

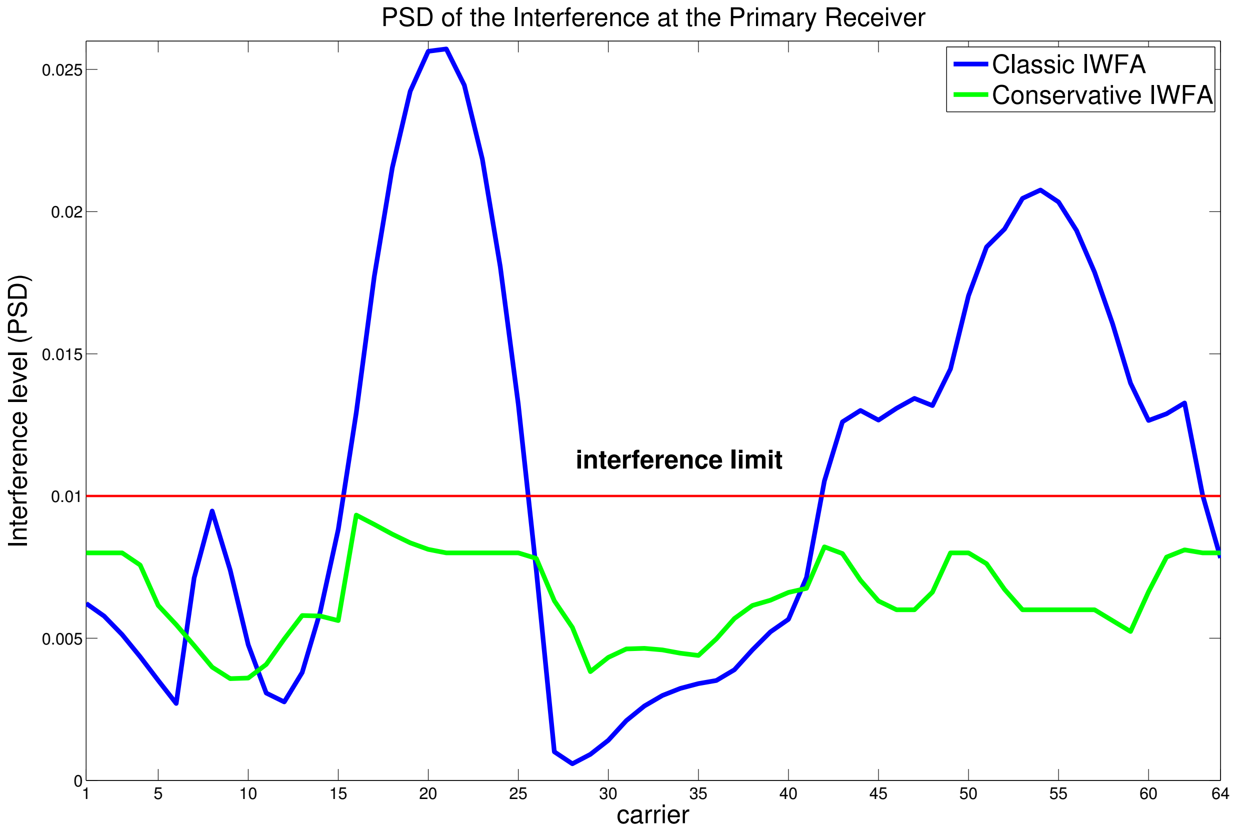
$$\left|H_{qp}^{(P,S)}(k)\right|^2 p_q(k) \leq I_{pq}(k), \quad k = 1, \ldots, N \quad p = 1, \ldots, P$$

  - $|H_{qp}^{(P,S)}(k)|^2$ is the cross-channel gain between the qth secondary and the pth primary user
  - $I_{pq}(k)$ is the maximum level of interference tolerable by the primary user p from the secondary user q over the subchannel k.

- Equivalently, we can write these constraints in terms of mask constraints

$$p_q(k) \leq p_q^{\max}(k) \triangleq \min_{p=1,\ldots,P} \frac{I_{pq}(k)}{\left|H_{qp}^{(P,S)}(k)\right|^2} \quad k = 1, \ldots, N.$$

- The interference constraints are now satisfied:



- However... this method may be too conservative as the level of interference from each secondary user is limited *individually* in a conservative way.

Revised Signal Model for CR Systems

- The really important quantity is not the individual interference generated by each secondary user but the *aggregate interference* generated by all of them.

- We can then limit the aggregate interference:
  - per-carrier constraints:

$$\sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \leq I_p(k), \quad p = 1, \ldots, P, \quad k = 1, \ldots, N$$

  - interference-temperature limits

$$\sum_{k=1}^{N} \sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \leq I_p(k), \quad p = 1, \ldots, P$$

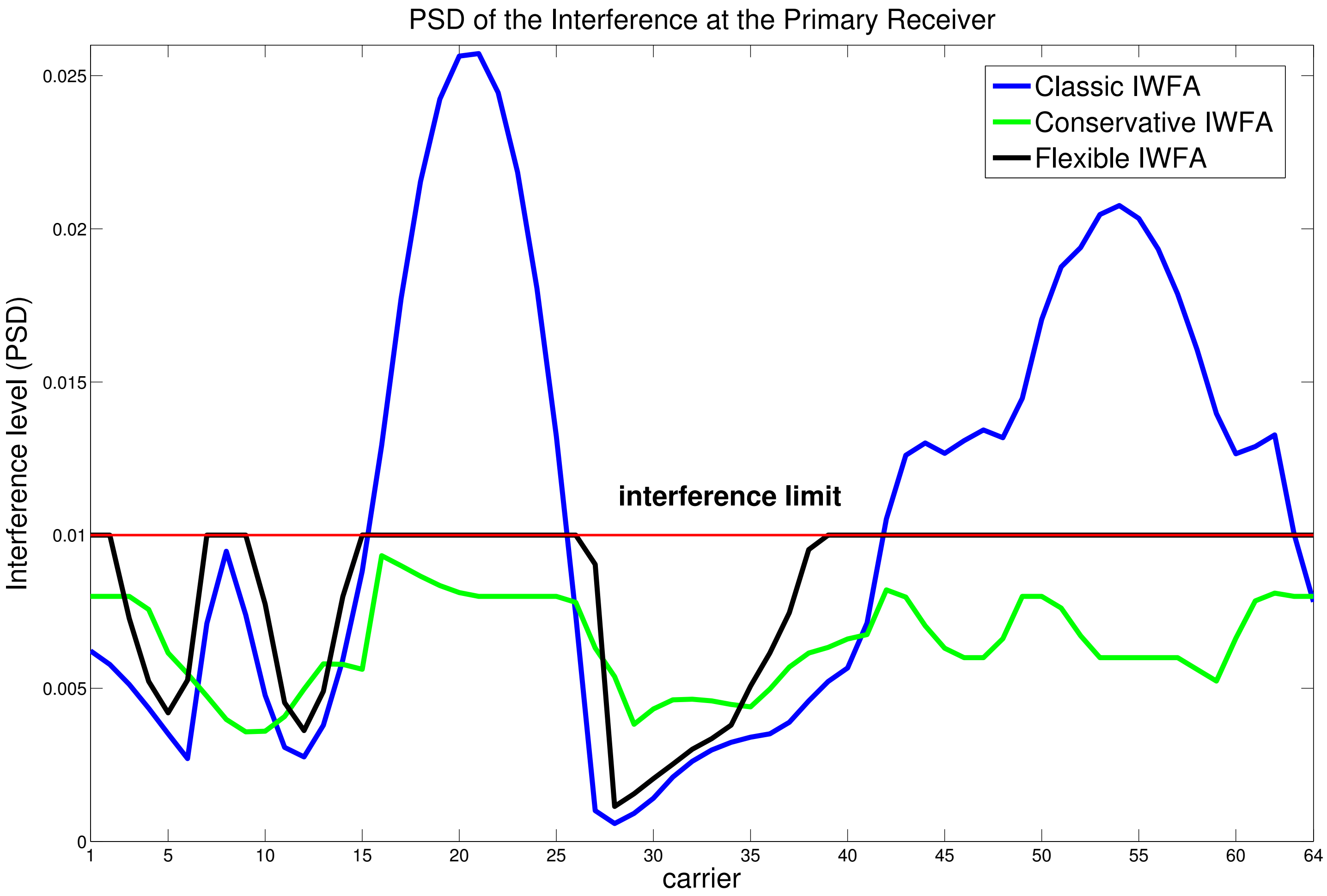
CR Game with Coupling Constraint

- Proposed game theoretical formulation [Pan-Scu-Pal-Fac'TSP10]:

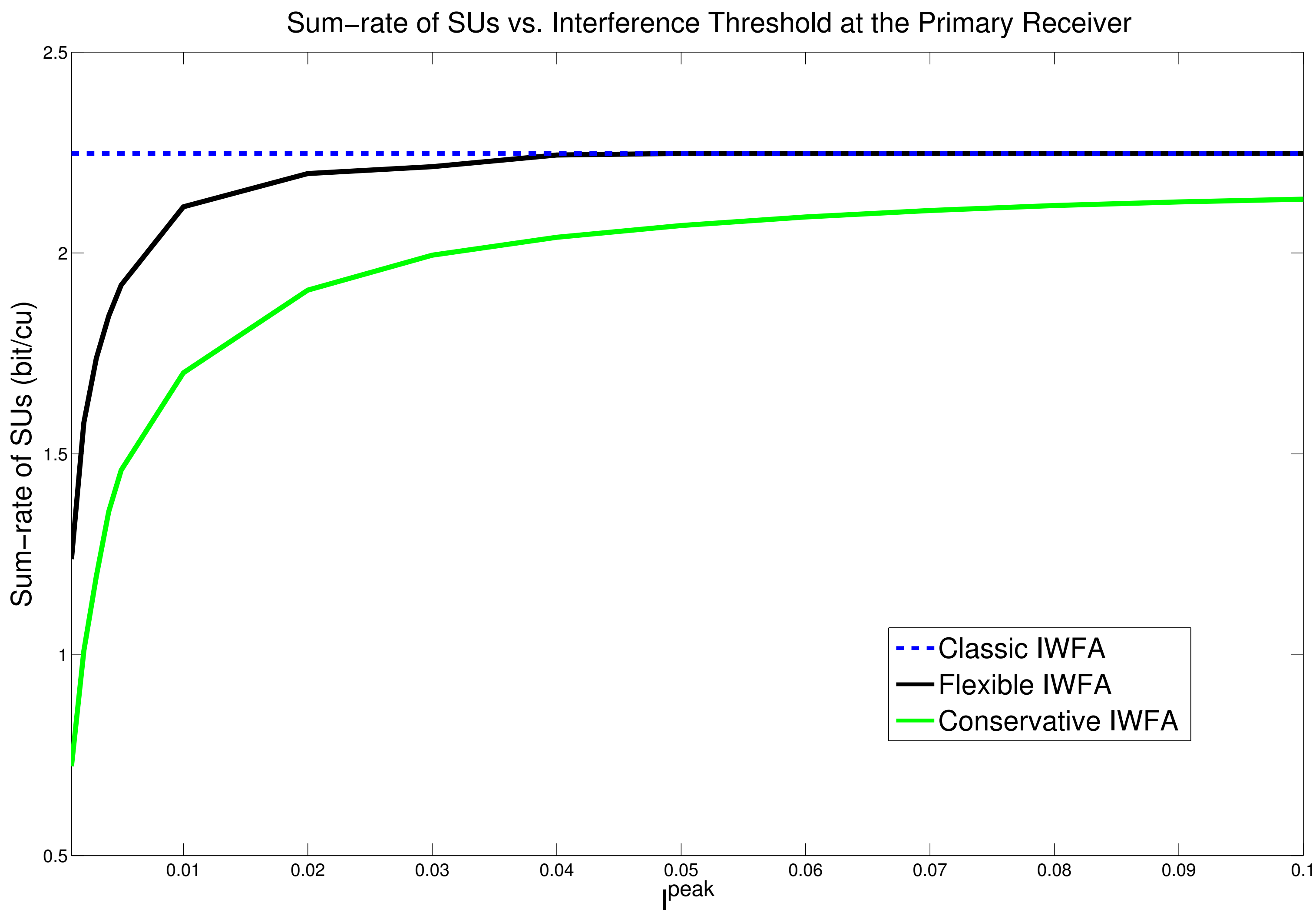
$$\mathcal{G}: \quad \begin{array}{ll} \underset{\mathbf{p}_q \geq \mathbf{0}}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_q(k)\right) \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q \end{array} \qquad \forall q = 1, \ldots, Q$$

$$\sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \leq I_p(k), \quad p = 1, \ldots, P, \qquad k = 1, \ldots, N.$$

- Indeed, this new reformulation achieves our goal without accidentally violating the limit:



or being too conservative:



How to Deal with the Coupling Constraint

- The price to pay for including the coupling constraints is twofold:
  - on a mathematical level, it complicates the analysis of the game and its design $\Rightarrow$ no previous GT results can be used
  - on the practical side, this new game must include some mechanism to calculate the aggregate interference.

- For the mathematical analysis and design, we need more advance tools: VI theory for the analysis of GNEP with shared constraints.

Formulation of the Game with Coupling Constraint

- Recall the game formulation with the coupling constraint [Pan-Scu-Pal-Fac'TSP10]:

$$\mathcal{G}: \quad \begin{array}{ll} \underset{\mathbf{p}_q \geq \mathbf{0}}{\text{maximize}} & r_q(\mathbf{p}_q, \mathbf{p}_{-q}) \triangleq \sum_{k=1}^{N} \log\left(1 + \text{sinr}_q(k)\right) \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q \end{array} \qquad \forall q = 1, \ldots, Q$$

$$\sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \leq I_p(k), \quad p = 1, \ldots, P, \qquad k = 1, \ldots, N.$$

- This is a GNEP with shared constraints

- It can be "rewritten" as a VI problem (caveat: only the variational solutions with common multipliers are considered).

Formulation of the Game with Coupling Constraint via Pricing

- Consider a game with pricing:

$$\mathcal{G}_{\boldsymbol{\lambda}}: \quad \begin{array}{ll} \underset{\mathbf{p}_q \geq \mathbf{0}}{\text{maximize}} & r_q(\mathbf{p}_q, \mathbf{p}_{-q}) - \sum_{p=1}^{P} \sum_{k=1}^{N} \lambda_{p,k} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q \end{array} \quad \forall q$$

where the prices $\lambda_{p,k} \geq 0$ are chosen such that

$$(\text{CC}): \qquad 0 \leq \lambda_{p,k} \quad \perp \quad I_p(k) - \sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \geq 0 \qquad \forall p, \quad \forall k$$

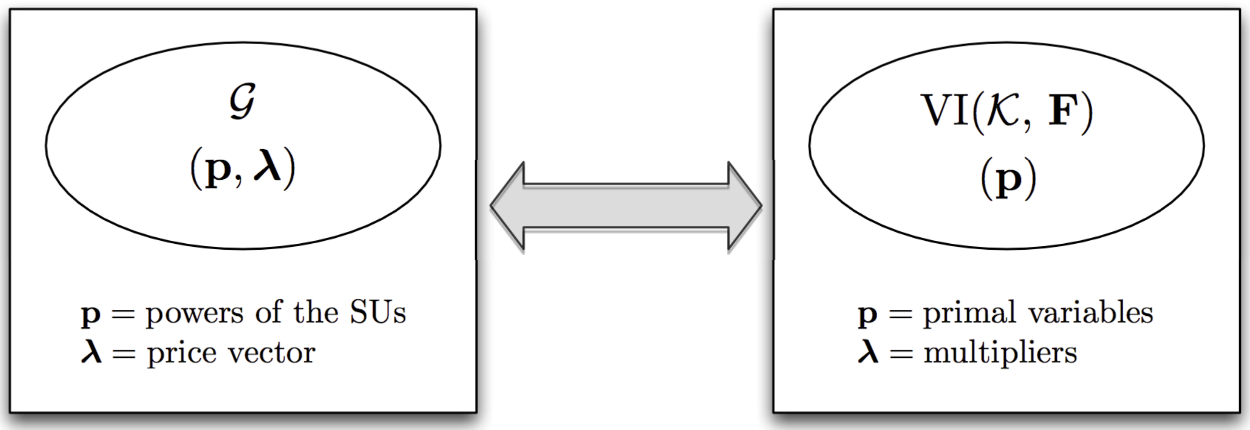
- Parametric NEP & CC outside: classical GT fails.
- We will now rewrite the game $\mathcal{G} \triangleq \mathcal{G}_{\boldsymbol{\lambda}} \cup (\text{CC})$ as a VI problem.

- **Theorem (Game as a VI) [Pan-Scu-Pal-Fac'TSP10]:** *The game* $\mathcal{G}$ *is equivalent to the* VI$(\mathcal{K}, \mathbf{F})$ *where*

$$\mathcal{K} \triangleq \left\{ \mathbf{p} \in \mathbb{R}_{+}^{NQ} \,:\, \sum_{k=1}^{N} p_q(k) \leq P_q \text{ and } \sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q(k) \leq I_p(k), \ \forall\, q,\, p\,, k \right\}$$

and

$$\mathbf{F}(\mathbf{p}) \triangleq \begin{pmatrix} -\nabla_{\mathbf{p}_1} r_1(\mathbf{p}) \\ \vdots \\ -\nabla r_{\mathbf{p}_Q}(\mathbf{p}) \end{pmatrix}$$



Results from the VI Framework: Solution Analysis

- Using the VI framework we can study existence/uniqueness of the solution and devise distributed algorithms.

- **Theorem (Existence and Uniqueness of the NE of $\mathcal{G}$) [Pan-Scu-Pal-Fac,TSP10]:**
  - *The* VI$(\mathcal{K}, \mathbf{F})$ *always admits a solution* $\mathbf{p}^{\mathrm{VI}}$ (the NE of $\mathcal{G}$)
  - *If* $\boldsymbol{\Upsilon} \succ \mathbf{0}$, *then* $\mathbf{p}^{\mathrm{VI}}$ *is unique.*

- We can now devise algorithms based on the VI, but they will not be distributed due to the coupling in $\mathcal{K}$.

- But we really want distributed algorithms...

Toward Distributed Algs: A NCP Reformulation

- Let's introduce now the interference violation function $\boldsymbol{\Phi}(\boldsymbol{\lambda}) : \mathbb{R}_{+}^{PN} \ni \boldsymbol{\lambda} \mapsto \mathbb{R}^{PN}$

$$\boldsymbol{\Phi} : \boldsymbol{\lambda} \mapsto \left(\left(I_p(k) - \sum_{q=1}^{Q} \left| H_{qp}^{(P,S)}(k) \right|^2 p_q^{\star}(k; \boldsymbol{\lambda}) \right)_{k=1}^{N} \right)_{p=1}^{P} \tag{1}$$

where $\mathbf{p}^{\star}(\boldsymbol{\lambda})$ is a Nash equilibrium of $\mathcal{G}_{\boldsymbol{\lambda}}$ for a *given* $\boldsymbol{\lambda}$

- **Theorem (Game as NCP) [Pan-Scu-Pal-Fac'TSP10]:** *If* $\boldsymbol{\Upsilon} \succ \mathbf{0}$, *then* $\mathcal{G}$ *is equivalent to the NCP in the price* $\boldsymbol{\lambda}$

$$\text{NCP}(\boldsymbol{\Phi}) : \qquad 0 \leq \boldsymbol{\lambda} \perp \boldsymbol{\Phi}(\boldsymbol{\lambda}) \geq 0 \quad \Leftrightarrow \quad \text{VI}\left(\mathbb{R}_{+}^{PN}, \boldsymbol{\Phi}\right).$$

- The NCP reformulation is instrumental to devise distributed algorithms.

Distributed Algorithms based on NCP

Algorithm 4: Projection algorithm with constant step-size

(S.0) : Choose any $\boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$, and the step size $\tau > 0$, and set $n = 0$

(S.1) : If $\boldsymbol{\lambda}^{(n)}$ satisfies a suitable termination criterion: STOP

(S.2) : Given $\boldsymbol{\lambda}^{(n)}$, compute $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ as the NE solution of the NEP $\mathcal{G}_{\boldsymbol{\lambda}}$ with *fixed* prices $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(n)}$

(S.3) : Update the price vectors:

$$\boldsymbol{\lambda}^{(n+1)} = \left[\boldsymbol{\lambda}^{(n)} - \tau\, \boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)\right]^{+} \qquad (2)$$

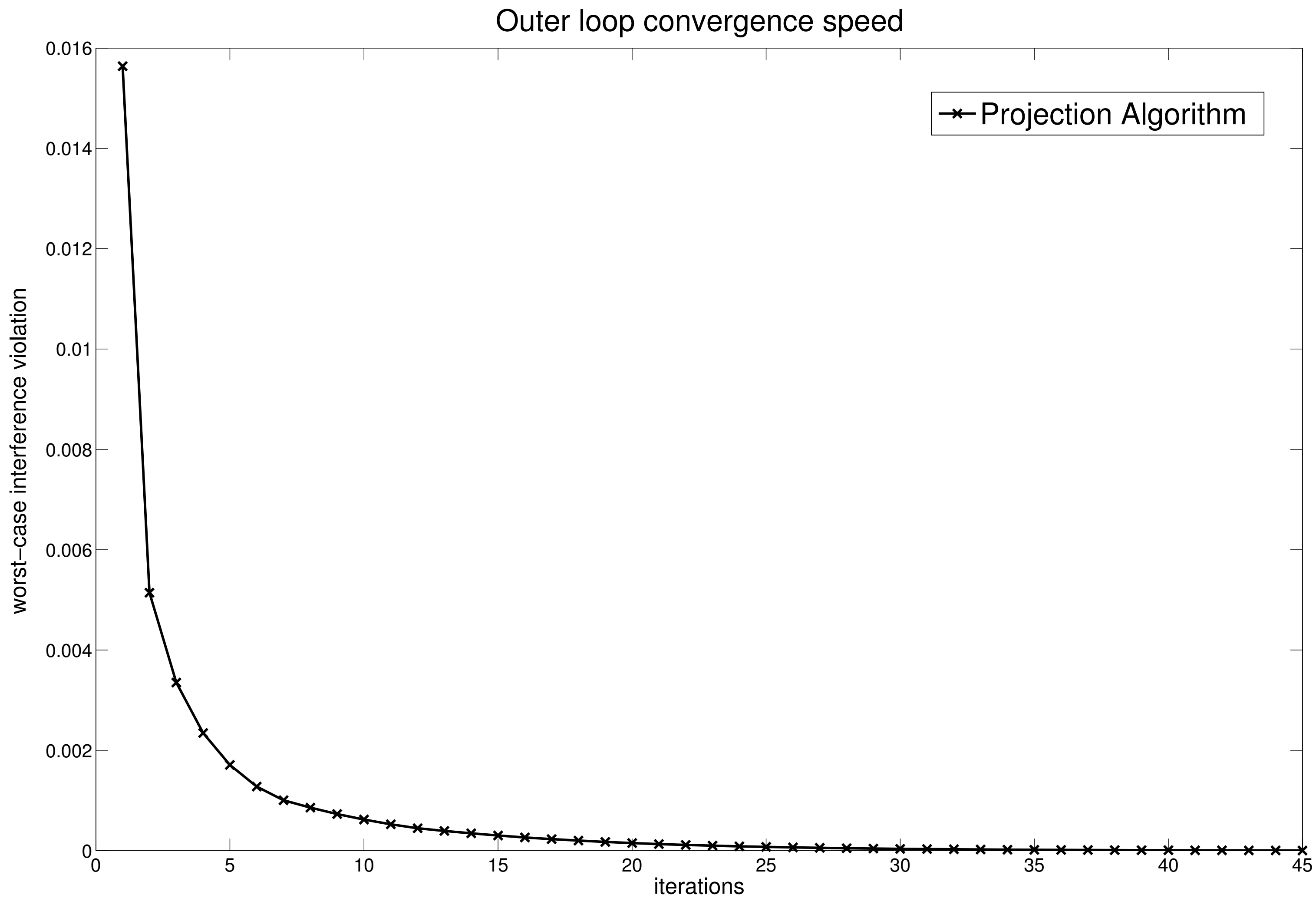
(S.4) : Set $n \leftarrow n + 1$; go to (S.1)

- **Theorem (Global convergence) [Pan-Scu-Pal-Fac,TSP10]:** *If* $\boldsymbol{\Upsilon} \succ \mathbf{0}$ and *the step-size* τ *is sufficiently small, then the sequence* $\{\boldsymbol{\lambda}^{(n)}\}_{n=0}^{\infty}$ *generated by the algorithm converges to a solution of the* NCP($\boldsymbol{\Phi}$).

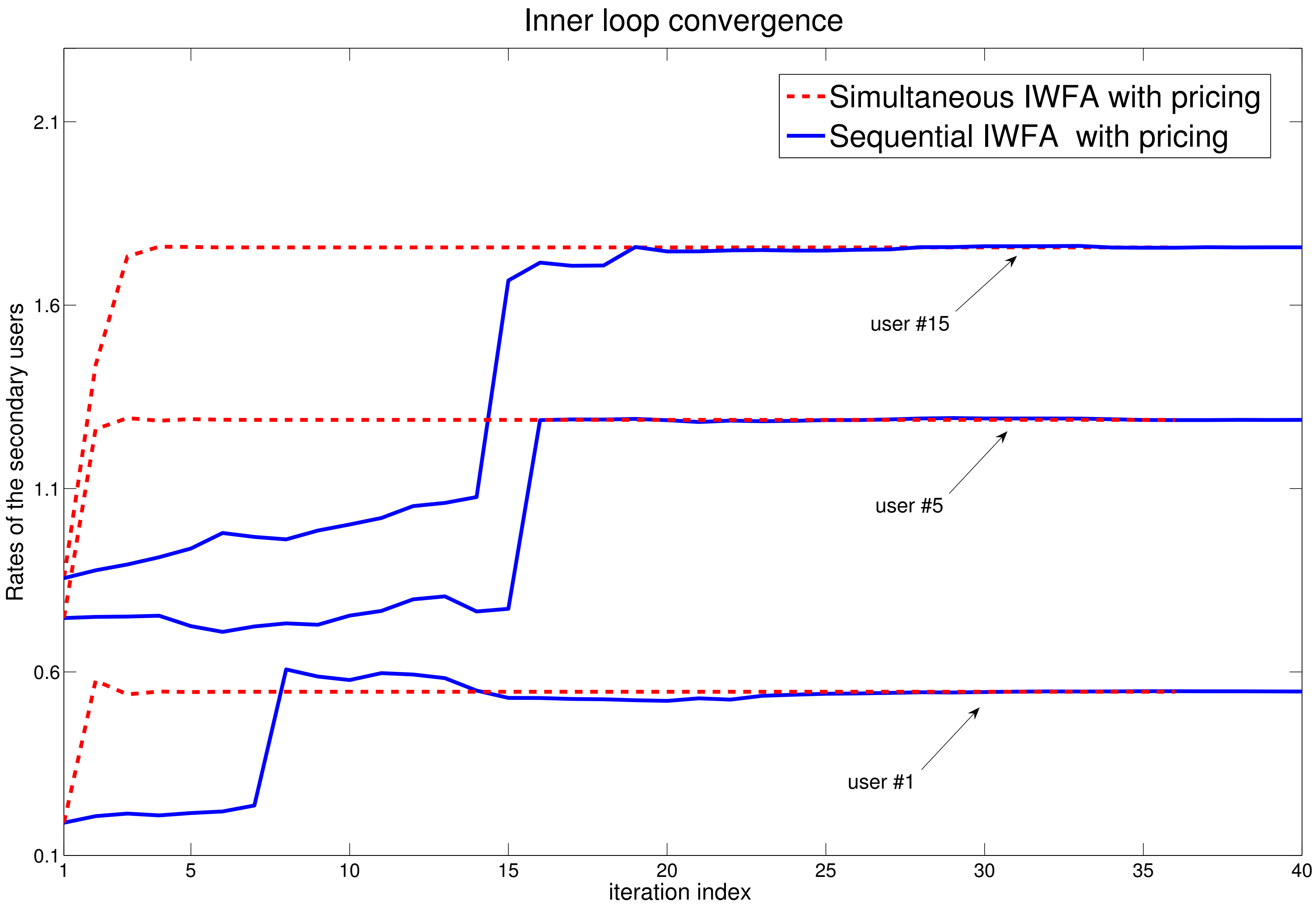
- Distributed implementation (limited signaling):
  - **Inner loop**: The NE $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ of $\mathcal{G}_{\boldsymbol{\lambda}^{(n)}}$ can be computed using the asynchronous IWFA (convergence is guaranteed under $\boldsymbol{\Upsilon} \succ \mathbf{0}$)
  - **Outer Loop** (*Spectrum leasing CR model*): at the iteration n, the PUs measure the interference violation $\boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)$, update the prices $\boldsymbol{\lambda}^{(n+1)}$ via the projection (2), and broadcast $\boldsymbol{\lambda}^{(n+1)}$ to the SUs who play the game $\mathcal{G}_{\boldsymbol{\lambda}^{(n+1)}}$
  - **Outer Loop** (*Common CR model*): the SUs update the prices as well by estimating the interference violation $\boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)$ via *consensus algorithms.*

- Several other algorithms have been considered that differ in the trade-off between SUs/PUs signaling, computational complexity, convergence conditions [Pan-Scu-Pal-Fac,TSP10]

Convergence of Outer Loop



Convergence of Inner Loop



Summary

Summary

- Theory: We have developed a fairly general mathematical framework based on VI suitable to study general equilibrium problems and design distributed algorithms.

- Applications: Using VI as a framework we have considered and solved a variety of game formulations in communications such as wireless ad-hoc networks, cognitive radio systems, network flow control problems, robust networks .

- Take-home message: Variational Inequality theory is a perfect mathematical framework for the analysis and design of fairly general equilibrium problems, both in theory and practice.

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End of Talk

Thank you !!

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