Introduction to Convex Optimization

Prof. Daniel P. Palomar

The Hong Kong University of Science and Technology (HKUST)

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Outline

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Optimization Problem

- General optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

  where

  $x = (x_1, \ldots, x_n)$ is the optimization variable
  $f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ is the objective function
  $f_i : \mathbf{R}^n \longrightarrow \mathbf{R}, \quad i = 1, \ldots, m$ are inequality constraint functions
  $h_i : \mathbf{R}^n \longrightarrow \mathbf{R}, \quad i = 1, \ldots, p$ are equality constraint functions.

- **Goal**: find an optimal solution $x^\star$ that minimizes f_0 while satisfying all the constraints.

Examples

Convex optimization is currently used in many different areas:

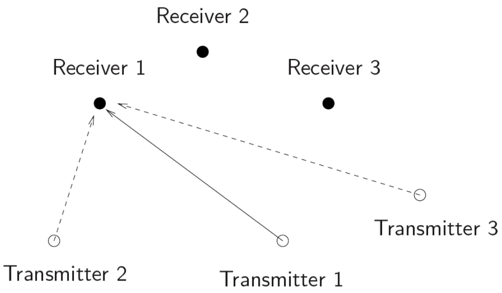
- circuit design (start-up named Barcelona in Silicon Valley)
- signal processing (e.g., filter design)
- communication systems (e.g., transceiver design, beamforming design, ML detection, power control in wireless)
- financial engineering (e.g., portflio design, index tracking)
- image proc. (e.g., deblurring, compressive sensing, blind separation)
- machine learning
- biomedical applications (e.g., analysis of DNA)

Examples: Elements in the Formulation

- An optimization problem has three basic elements: 1) variables, 2) constraints, and 3) objective.
- Example: device sizing in electronic circuits:
  - variables: device widths and lengths
  - constraints: manufacturing limits, timing requirements, max area
  - objective: power consumption
- Example: portfolio optimization:
  - variables: amounts invested in different assets
  - constraints: budget, max investments per asset, min return
  - objective: overall risk or return variance.

Example: Power Control in Wireless Networks

- Consider a wireless network with n logical transmitter/receiver pairs:



- Goal: design the power allocation so that each receiver receives minimum interference from the other links.

Example: Power Control in Wireless Networks

- The signal-to-inerference-plus-noise-ratio (SINR) at the ith receiver is

$$\mathrm{sinr}_i = \frac{p_i G_{ii}}{\sum_{j \neq i} p_j G_{ij} + \sigma_i^2}$$

where

p_i is the power used by the ith transmitter
G_{ij} is the path gain from transmitter j to receiver i
σ_i^2 is the noise power at the ith receiver.

- **Problem**: maximize the weakest SINR subject to power constraints $0 \leq p_i \leq p_i^{\max}$:

$$\begin{array}{ll} \underset{\mathbf{p}}{\text{maximize}} & \min_{i=1,\ldots,n} \frac{p_i G_{ii}}{\sum_{j \neq i} p_j G_{ij} + \sigma_i^2} \\ \text{subject to} & 0 \leq p_i \leq p_i^{\max} \qquad i = 1, \ldots, n. \end{array}$$

Solving Optimization Problems

- General optimization problems are very difficult to solve (either long computation time or not finding the best solution).
- Exceptions: least-squares problems, linear programming problems, and convex optimization problems.
- **Least-squares (LS)**:

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2$$

  - solving LS problems: closed-form solution $x^\star = \left(A^T A\right)^{-1} A^T b$ for which there are reliable and efficient algorithms; mature technology
  - using LS: easy to recognize

Solving Optimization Problems

- **Linear Programming (LP)**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - solving LP problems: no closed-form solution, but reliable and efficient algorithms and software; mature technology
  - using LP: not as easy to recognize as LS problems, a few standard tricks to convert problems into LPs

- **Convex optimization**:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  - solving convex problems: no closed-form solution, but still reliable and efficient algorithms and software; almost a technology
  - using convex optimization: often difficult to recognize, many tricks for transforming problems into convex form.

Nonconvex Optimization

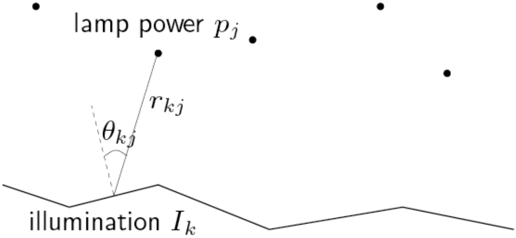
Nonconvex optimization problems are generally very difficult to solve, although there are some rare exceptions.

In general, they require either a long computation time or the compromise of not always finding the optimal solution:

- local optimization: fast algorithms, but no guarantee of global optimality, only local solution around the initial point
- global optimization: worst-case complexity grows exponentially with problem size, but finds global solution.

Example: Lamp Illumination Problem

- Consider m lamps illuminating n small flat patches:



- Goal: achieve a desired illumination I_{des} on all patches with bounded lamp powers.

Example: Lamp Illumination Problem

- The intensity I_k at patch k depends linearly on the lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{kj} p_j$$

where the coefficients a_{kj} are given by $a_{kj} = \cos\theta_{kj}/r_{kj}^2$.

- Problem formulation: since the illumination is perceived logarithmically by the eye, a good formulation of the problem is

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k |\log I_k - \log I_{\text{des}}| \\ \text{subject to} & 0 \le p_j \le p_{\max}, \quad j = 1,\ldots,m \\ & I_k = \sum_{j=1}^{m} a_{kj} p_j, \quad k = 1,\ldots,n. \end{array}$$

- How to solve the problem? The answer is: it depends on how much you know about optimization.

Example: Lamp Illumination Problem

- If you don't know anything, then you just take a heuristic guess like using a uniform power $p_j = p$, perhaps trying different values of p.
- If you know about least-squares, then approximate the problem as

$$\underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} \quad \sum_{k=1}^{n} \left(I_k - I_{\text{des}}\right)^2$$

and then round p_j if $p_j > p_{\max}$ or $p_j < 0$.

- If you know about linear programming, then approximate it as

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k \left|I_k - I_{\text{des}}\right| \\ \text{subject to} & 0 \leq p_j \leq p_{\max}, \quad j = 1, \ldots, m. \end{array}$$

- If you know about convex optimization, after staring at the problem long enough, you may realize that you can reformulate it in convex form:

$$\begin{array}{ll} \underset{I_1,\ldots,I_n,p_1,\ldots,p_m}{\text{minimize}} & \max_k h\left(I_k / I_{\text{des}}\right) \\ \text{subject to} & 0 \leq p_j \leq p_{\max}, \quad j = 1, \ldots, m. \end{array}$$

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Example: Lamp Illumination Problem

- **Additional constraints**: does adding the constraints below complicate the problem?

  (A) no more than half of total power is in any 10 lamps
  (B) no more than half of the lamps are on ($p_j > 0$).

- Answer: adding (a) does not complicate the problem, whereas adding (b) makes the problem extremely difficult.
- Moral: untrained intuition doesn't always work; one needs to obtain the proper background and develop the right intuition to discern between difficult and easy problems.

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History Snapshop of Convex Optimization

- **Theory** (convex analysis): ca1900-1970 (e.g. Rockafellar)
- **Algorithms**:
  - 1947: simplex algorithm for linear programming (Dantzig)
  - 1960s: early interior-point methods (Fiacco & McCormick, Dikin)
  - 1970s: ellipsoid method and other subgradient methods
  - 1980s: polynomial-time interior-point methods for linear programming (Karmakar 1984)
  - late 1980s-now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)
- **Applications**:
  - before 1990s: mostly in operations research; few in engineering
  - since 1990: many new applications in engineering and new problem classes (SDP, SOCP, robust optim.)

References on Convex Optimization

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

  https://web.stanford.edu/~boyd/cvxbook/

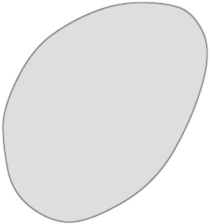
- Daniel P. Palomar and Yonina C. Eldar, Eds., *Convex Optimization in Signal Processing and Communications*, Cambridge University Press, 2009.
- Ben Tal & Nemirovsky, *Lectures on Modern Convex Optimization*. SIAM 2001.
- Nesterov & Nemirovsky, *Interior-point Polynomial Algorithms in Convex Programming*. SIAM 1994.

Outline

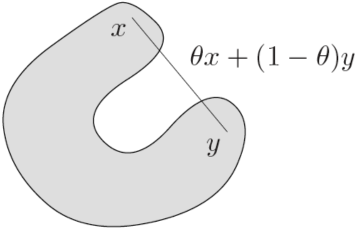
Definition of Convex Set

- A set $C \in \mathbf{R}^n$ is said to be **convex** if the line segment between any two points is in the set: for any $x, y \in C$ and $0 \leq \theta \leq 1$,

$$\theta x + (1 - \theta)\, y \in C.$$



convex



non-convex

Examples: Hyperplanes and Halfspaces

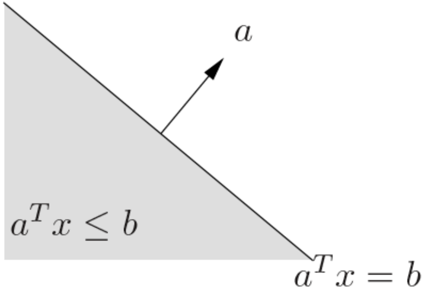
- **Hyperplane**:

$$C = \left\{ x \mid a^T x = b \right\}$$

where $a \in \mathbf{R}^n$, $b \in \mathbf{R}$.

- **Halfspace**:

$$C = \left\{ x \mid a^T x \leq b \right\}$$

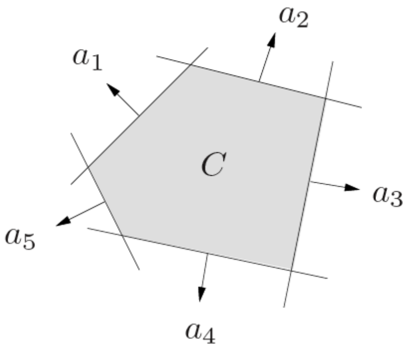


Example: Polyhedra

- **Polyhedron**:

$$C = \{x \mid Ax \leq b,\ Cx = d\}$$

where $A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^m$, $d \in \mathbf{R}^p$.



Examples: Euclidean Balls and Ellipsoids

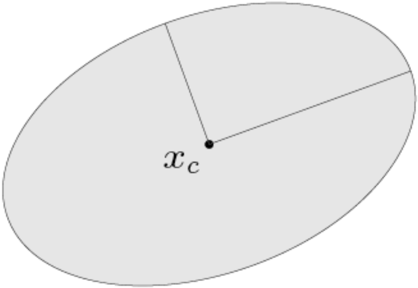
- **Euclidean ball** with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}.$$

- **Ellipsoid**:

$$E(x_c, P) = \left\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\right\} = \{x_c + Au \mid \|u\|_2 \leq 1\}$$

with $P \in \mathbf{R}^{n\times n} \succ 0$ (positive definite).



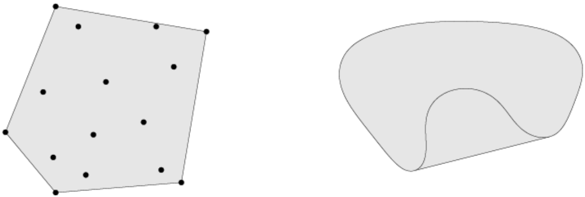
Convex Combination and Convex Hull

- **Convex combination** of $x_1, \ldots, x_k$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

  with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$.

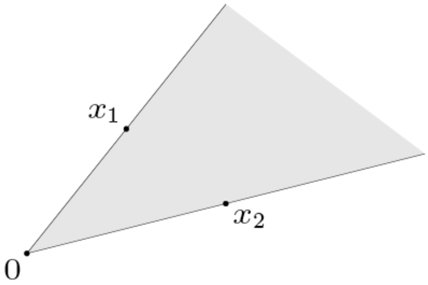
- **Convex hull** of a set: set of all convex combinations of points in the set.



Convex Cones

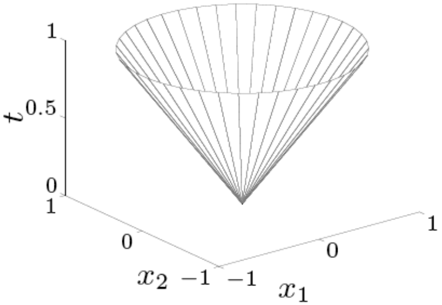
- A set $C \in \mathbf{R}^n$ is said to be a **convex cone** if the ray from each point in the set is in the set: for any $x_1, x_2 \in C$ and $\theta_1,\ \theta_2 \geq 0$,

$$\theta_1 x_1 + \theta_2 x_2 \in C.$$



Norm Balls and Norm Cones

- **Norm ball** with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$ where $\|\cdot\|$ is a norm.
- **Norm cone**: $\left\{(x, t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\right\}$.
- Euclidean norm cone or second-order cone (aka ice-cream cone):

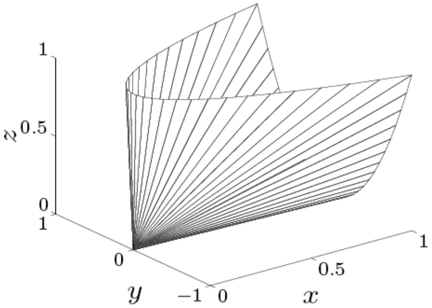


Positive Semidefinite Cone

- **Positive semidefinite (PSD) cone**:

$$\mathbf{S}_{+}^{n} = \left\{ X \in \mathbf{R}^{n \times n} \mid X = X^{T} \succeq 0 \right\}.$$

- Example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_{+}^{2}$



Operations that Preserve Convexity

How do we establish the convexity of a given set?

1. Applying the definition:

$$x, y \in C,\ 0 \leq \theta \leq 1 \Longrightarrow \theta x + (1 - \theta)\, y \in C$$

   which can be cumbersome.

2. Showing that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, etc.) by operations that preserve convexity:
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

- **Intersection**: if S_1, $S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex.
- Example: a polyhedron is the intersection of halfspaces and hyperplanes.
- Example:

$$S = \{x \in \mathbf{R}^n \mid |p_x(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p_x(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_n \cos nt$.

Affine Function

- **Affine composition**: the image (and inverse image) of a convex set under an affine function $f(x) = Ax + b$ is convex:

$$S \subseteq \mathbf{R}^n \text{ convex} \Longrightarrow f(S) = \{f(x) \mid x \in S\} \text{ convex.}$$

- Examples: scaling, translation, projection.
- Example: $\left\{(x,t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\right\}$ is convex, so is

$$\left\{x \in \mathbf{R}^n \mid \|Ax + b\| \leq c^T x + d\right\}.$$

- Example: solution set of LMI: $\{x \in \mathbf{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$.

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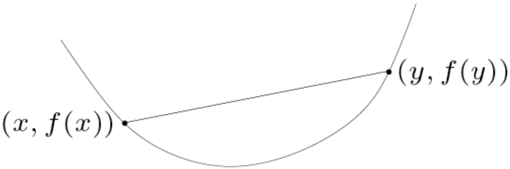
Outline

Definition of Convex Function

- A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is said to be **convex** if the domain, $\mathbf{dom}\, f$, is convex and for any $x, y \in \mathbf{dom}\, f$ and $0 \leq \theta \leq 1$,

$$f\left(\theta x + (1-\theta)\, y\right) \leq \theta f\left(x\right) + (1-\theta)\, f\left(y\right).$$



- f is strictly convex if the inequality is strict for $0 < \theta < 1$.
- f is concave if $-f$ is convex.

Examples on R

Convex functions:

- affine: $ax + b$ on $\mathbf{R}$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$ (e.g., $|x|$)
- powers: x^p on $\mathbf{R}_{++}$, for $p \geq 1$ or $p \leq 0$ (e.g., x^2)
- exponential: e^{ax} on $\mathbf{R}$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

Concave functions:

- affine: $ax + b$ on $\mathbf{R}$
- powers: x^p on $\mathbf{R}_{++}$, for $0 \leq p \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

Examples on $\mathbf{R}^n$

- **Affine functions** $f(x) = a^T x + b$ are convex and concave on $\mathbf{R}^n$.
- **Norms** $\|x\|$ are convex on $\mathbf{R}^n$ (e.g., $\|x\|_\infty$, $\|x\|_1$, $\|x\|_2$).
- **Quadratic functions** $f(x) = x^T P x + 2q^T x + r$ are convex $\mathbf{R}^n$ if and only if $P \succeq 0$.
- The **geometric mean** $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbf{R}^n_{++}$.
- The **log-sum-exp** $f(x) = \log \sum_i e^{x_i}$ is convex on $\mathbf{R}^n$ (it can be used to approximate $\max_{i=1,\ldots,n} x_i$).
- **Quadratic over linear**: $f(x, y) = x^2/y$ is convex on $\mathbf{R}^n \times \mathbf{R}_{++}$.

Examples on $\mathbf{R}^{n\times n}$

- **Affine functions**: (prove it!)

$$f(X) = \mathrm{Tr}(AX) + b$$

  are convex and concave on $\mathbf{R}^{n\times n}$.

- **Logarithmic determinant function**: (prove it!)

$$f(X) = \mathrm{logdet}(X)$$

  is concave on $\mathbf{S}^n = \{X \in \mathbf{R}^{n\times n} \mid X \succeq 0\}$.

- **Maximum eigenvalue function**: (prove it!)

$$f(X) = \lambda_{\max}(X) = \sup_{y\neq 0} \frac{y^T X y}{y^T y}$$

  is convex on $\mathbf{S}^n$.

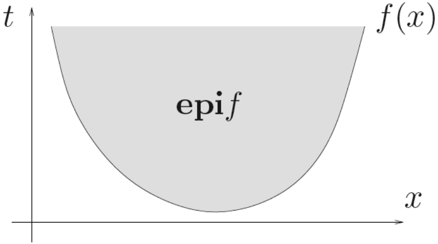
Epigraph

- The **epigraph** of f if the set

$$\text{epi}\, f = \left\{(x,t) \in \mathbf{R}^{n+1} \mid x \in \text{dom}\, f,\ f(x) \le t\right\}.$$

- Relation between convexity in sets and convexity in functions:

$$f \text{ is convex} \iff \text{epi}\, f \text{ is convex}$$



Restriction of a Convex Function to a Line

- $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is convex if and only if the function $g : \mathbf{R} \longrightarrow \mathbf{R}$

$$g(t) = f(x + tv), \qquad \mathrm{dom}\, g = \{t \mid x + tv \in \mathrm{dom}\, f\}$$

  is convex for any $x \in \mathrm{dom}\, f$, $v \in \mathbf{R}^n$.
- In words: a function is convex if and only if it is convex when restricted to an arbitrary line.
- Implication: we can check convexity of f by checking convexity of functions of one variable!
- Example: concavity of $\log\det(X)$ follows from concavity of $\log(x)$.

Restriction of a Convex Function to a Line

Example: concavity of $\operatorname{logdet}(X)$:

$$\begin{aligned} g(t) = \operatorname{logdet}(X + tV) &= \operatorname{logdet}(X) + \operatorname{logdet}\left(I + tX^{-1/2}VX^{-1/2}\right) \\ &= \operatorname{logdet}(X) + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

where λ_i's are the eigenvalues of $X^{-1/2}VX^{-1/2}$.
The function g is concave in t for any choice of $X \succ 0$ and V; therefore, f is concave.

First and Second Order Condition

- **Gradient** (for differentiable f):

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix}^T \in \mathbf{R}^n.$$

- **Hessian** (for twice differentiable f):

$$\nabla^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{ij} \in \mathbf{R}^{n \times n}.$$

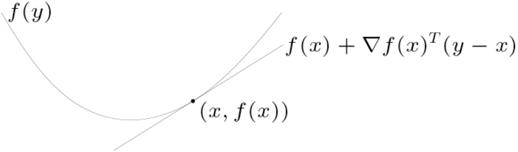
- Taylor series:

$$f(x+\delta) = f(x) + \nabla f(x)^T \delta + \frac{1}{2}\delta^T \nabla^2 f(x)\,\delta + o\left(\|\delta\|^2\right).$$

First and Second Order Condition

- **First-order condition**: a differentiable f with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall x, y \in \operatorname{dom} f$$



- Interpretation: first-order approximation if a global underestimator.
- **Second-order condition**: a twice differentiable f with convex domain is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \forall x \in \operatorname{dom} f$$

Examples

- **Quadratic function**: $f(x) = (1/2)\,x^T Px + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

  is convex if $P \succeq 0$.

- **Least-squares objective**: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

  is convex.

- **Quadratic-over-linear**: $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y & -x \end{bmatrix} \succeq 0$$

  is convex for $y > 0$.

Operations that Preserve Convexity

How do we establish the convexity of a given function?

1. Applying the definition.
2. With first- or second-order conditions.
3. By restricting to a line.
4. Showing that the functions can be obtained from simple functions by operations that preserve convexity:
    - nonnegative weighted sum
    - composition with affine function (and other compositions)
    - pointwise maximum and supremum, minimization
    - perspective

Operations that Preserve Convexity

- **Nonnegative weighted sum**: if f_1, f_2 are convex, then $\alpha_1 f_1 + \alpha_2 f_2$ is convex, with $\alpha_1, \alpha_2 \geq 0$.
- **Composition with affine functions**: if f is convex, then $f(Ax + b)$ is convex (e.g., $\|y - Ax\|$ is convex, $\log\det\left(I + HXH^T\right)$ is concave).
- **Pointwise maximum**: if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1, \ldots, f_m\}$ is convex.

Example: sum of r largest components of $x \in \mathbf{R}^n$:
$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$ where $x_{[i]}$ is the ith largest component of x.
Proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$.

Operations that Preserve Convexity

- **Pointwise supremum**: if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then

  $$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

  is convex.

Example: distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|.$$

Example: maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{y \neq 0} \frac{y^T X y}{y^T y}.$$

Operations that Preserve Convexity

- **Composition with scalar functions**: let $g : \mathbf{R}^n \longrightarrow \mathbf{R}$ and $h : \mathbf{R} \longrightarrow \mathbf{R}$, then the function $f(x) = h(g(x))$ satisfies:

$$f(x) \text{ is convex if } \begin{array}{l} g \text{ convex, } h \text{ convex nondecreasing} \\ g \text{ concave, } h \text{ convex nonincreasing} \end{array}$$

- **Minimization**: if $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

  is convex (e.g., distance to a convex set).

Example: distance to a set C:

$$f(x) = \inf_{y \in C} \|x - y\|$$

is convex if C is convex.

References on Convex Functions

Chapter 3 of

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

https://web.stanford.edu/~boyd/cvxbook/

Outline

General Optimization Problem

Optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array}$$

$x \in \mathbf{R}^n$ is the optimization variable

$f_0 : \mathbf{R}^n \longrightarrow \mathbf{R}$ is the objective function

$f_i : \mathbf{R}^n \longrightarrow \mathbf{R},\ i = 1, \ldots, m$ are inequality constraint functions

$h_i : \mathbf{R}^n \longrightarrow \mathbf{R},\ i = 1, \ldots, p$ are equality constraint functions.

Convex Optimization Problem

Convex optimization problem in standard form:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

where $f_0, f_1, \ldots, f_m$ are convex and equality constraints are affine.

- **Local and global optima**: any locally optimal point of a convex problem is globally optimal.
- Most problems are not convex when formulated.
- Reformulating a problem in convex form is an art, there is no systematic way.

Convex Optimization Problem

To be continued...

Thanks

For more information visit:

https://www.danielppalomar.com