

Practical Algorithms for a Family of Waterfilling Solutions

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Abstract

Many engineering problems that can be formulated as constrained optimization problems result in solutions given by a waterfilling structure; the classical example being the capacity-achieving solution for a frequency-selective channel. For simple waterfillings solutions with a single waterlevel and a single constraint (typically, a power constraint), some algorithms have been proposed in the literature to compute the solutions numerically. However, some other optimization problems result in significantly more complicated waterfillings solutions that include multiple waterlevels and multiple constraints. For such cases, it may still be possible to obtain practical algorithms to evaluate the solutions numerically, but only after a painstaking inspection of the specific waterfilling structure. In addition, a unified view of the different types of waterfilling solutions and the corresponding practical algorithms is missing.

The purpose of this paper is twofold. On the one hand, it overviews the waterfilling results existing in the literature from a unified viewpoint. On the other hand, it bridges the gap between a wide family of waterfilling solutions and their efficient implementation in practice; to be more precise, it provides a practical algorithm to evaluate numerically a general waterfilling solution, which includes the currently existing waterfilling solutions and others that may possibly appear in future problems.

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1 Introduction

The well-known classical waterfilling solution solves the problem of maximizing the mutual information between the input and the output of a channel composed of several subchannels (such as a frequency-selective channel, a time-varying channel, or a set of parallel subchannels arising from the use of multiple antennas at both sides of the link) with a global power constraint at the transmitter [1, 2, 3]. This capacity-achieving solution has the visual interpretation of pouring water over a surface given by the inverse of the subchannel gains, hence the name waterfilling or waterpouring (c.f. Section 2).

The capacity-achieving waterfilling solution has been considered in many works to design efficient communication systems [4, 5, 6]. There are, however, other completely different problems that result in similar waterfilling solutions. In particular, when the transmitter and receiver are jointly designed for communications through multiple-input multiple-output (MIMO) channels, these type of solutions typically appear [7]. Perhaps the most popular of such problems is the minimization of the sum of the mean square errors (MSEs) (equivalently, the trace of the MSE matrix) of the different subchannels existing within a MIMO channel, resulting in a waterfilling solution [8, 9, 10, 6, 11, 12] (for frequency-selective single-input single-output (SISO) channels, similar solutions were obtained already in the sixties [13]). If, instead, the system is designed to minimize the determinant of the MSE matrix, the classical capacity-achieving waterfilling result is again obtained [14, 12] (this is due to the direct relation between the determinant of the MSE matrix and the mutual information [15]). The maximization of the minimum signal to interference-plus-noise ratio (SINR) among the subchannels also results in a waterfilling solution [7]. In [16], the minimization of the average bit error rate (BER) over a set of parallel subchannels was extensively treated obtaining a waterfilling result. In [17], minimum BER solutions were obtained for the case of imperfect channel knowledge at the transmitter, with a waterfilling power allocation. Recently, the problem of joint transmit-receive design to achieve minimum average BER in MIMO channels (with equal constellations) has been solved independently in [18] and [7], obtaining a solution that includes the same waterfilling structure as in the minimization of the trace of the MSE matrix.

The previous waterfilling solutions are very simple to evaluate because all of them have a single waterlevel and a power constraint. As a consequence, it is quite straightforward to compute them numerically in practice: since the solution is parameterized with a single waterlevel, the problem reduces to obtaining the waterlevel such that the power constraint is satisfied with equality. In order to find the exact value of the waterlevel, different algorithms have been proposed that can be classified into *iterative algorithms* and *exact algorithms*. The iterative algorithms are trivially implemented in practice and get close to the exact value as the number of iterations goes to infinity [13, 19, 10, 16]. The exact algorithms give the exact value of the solution in a finite number of loops or iterations [20, 21, 5, 11, 22].

As opposed to the aforementioned simple waterfilling solutions, other design criteria, such as the

minimization of the maximum of the BERs of the subchannels or the maximization of the harmonic mean of the SINRs of the subchannels, result in significantly more complicated waterfilling solutions with multiple waterlevels and multiple constraints (not just a simple power constraint) [7]. The minimum power design in a MIMO point-to-point communication system that satisfies a set of QoS requirements among the used subchannels also results in a waterfilling solution with multiple waterlevels and multiple constraints [23]. In such cases, it is not clear how to compute the numerical solution in practice, not even by adopting an iterative method; however, after a painstaking analysis of the specific structure of each waterfilling solution, it is still possible to obtain practical algorithms that give the numerical solution [24]. It is thus desirable to develop a general approach to deal with these complicated waterfilling solutions, as opposed to obtaining results tailored to each particular case.

This paper considers a general waterfilling solution (including multiple waterlevels and multiple constraints) and provides a general algorithm to compute numerically the solution in practice. In other words, the paper bridges the gap between a wide family of waterfilling solutions and their practical implementation. Such a framework, not only encompasses the currently existing waterfilling results and algorithms, but it may also serve as a basis to obtain practical algorithms in future problems with a similar structure. To be more precise, as long as a waterfilling solution fits within the general model considered in this paper, it suffices to use the general algorithm (known to give the exact numerical solution) and particularize it to the specific case at hand, as opposed to having to analyze the structure of the specific waterfilling solution to obtain an algorithm (and to prove that it gives the exact solution).

The paper is structured as follows. In Section 2, the signal model for the classical capacity-achieving waterfilling solution and for a completely general waterfilling solution are introduced. Section 3 overviews the existing algorithms for simple waterfilling solutions and Section 4 provides the main result of the paper: an algorithm to evaluate numerically a general multi-waterlevel multi-constrained waterfilling solution. Section 5 considers then the particularization of the general algorithm to specific examples of practical interest. Finally, in Section 6, the conclusions are drawn.

2 Waterfilling Solutions

The classical capacity-achieving waterfilling result is the solution to the following constrained optimization problem:

$$\begin{aligned} \max_{\{x_i\}} \quad & \sum_{i=1}^L \log(1 + x_i \lambda_i) \\ \text{s.t.} \quad & \sum_{i=1}^L x_i \leq P_T, \\ & x_i \geq 0, \quad 1 \leq i \leq L \end{aligned} \tag{1}$$

given by

$$x_i = (\mu - \lambda_i^{-1})^+ \quad 1 \leq i \leq L \tag{2}$$

where $(x)^+ \triangleq \max(0, x)$, L is the number of subchannels, λ_i is the gain of the i th subchannel, and μ is the waterlevel chosen to satisfy the power constraint with equality $\sum_i x_i = P_T$.

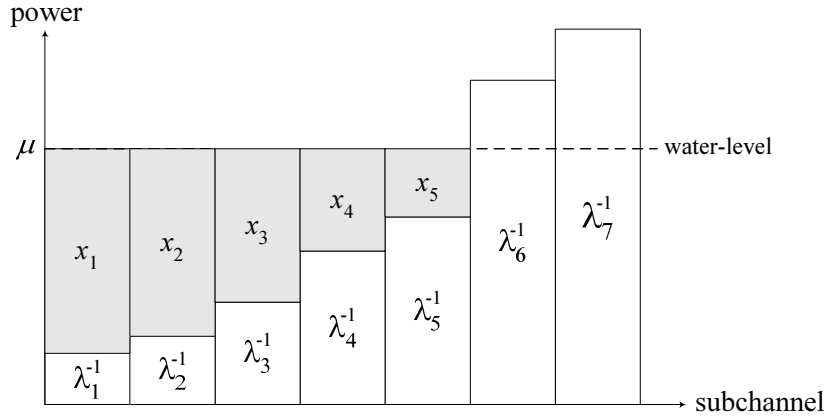


Figure 1: Illustration of the classical waterfilling power allocation $x_i = (\mu - \lambda_i^{-1})^+$ with $\sum_i x_i = P_T$.

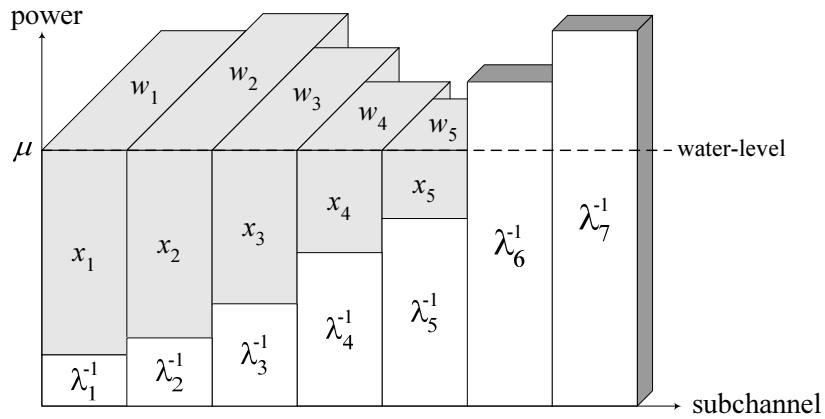


Figure 2: Illustration of the general waterfilling power allocation $x_i = (\mu - \lambda_i^{-1})^+$ with $\sum_i x_i w_i = P_T$.

As can be seen in Figure 1, this capacity-achieving solution has the visual interpretation of pouring water over a surface (or curve) given by the inverse of the subchannel gains, hence the name waterfilling or waterpouring [1, 2]. More general waterfilling expressions such as $x_i = (\mu a_i - b_i)^+$, where the a_i 's and b_i 's are arbitrary positive numbers, also have the same visual interpretation after the change of variables $\tilde{x}_i = x_i/a_i$, $\tilde{\lambda}_i = a_i/b_i$, and with a resulting weighted power constraint given by $\sum_i \tilde{x}_i w_i = P_T$, where $w_i = a_i$ are weights that can be visually interpreted as the width of each of the subchannels as can be observed from Figure 2.

As has been previously mentioned, there are many other constrained optimization problems that result in similar waterfilling solutions, possibly with multiple waterlevels and multiple constraints. It would be of a great interest to deal simultaneously with all these waterfilling problems by developing a general framework that embraces a wide range of waterfilling solutions.

A very general multi-waterlevel multi-constrained waterfilling result can be defined as any solution

of the form

$$x_{k,i} = (\mu_k a_{k,i} - b_{k,i})^+ \quad 1 \leq k \leq N, 1 \leq i \leq L_k \quad (3)$$

subject to a set of M arbitrary equality (inequality) constraints

$$f_m(\{\mu_k\}) = (\leq) 0 \quad 1 \leq m \leq M \quad (4)$$

where the μ_k 's are the N multiple waterlevels, the $x_{k,i}$'s are the variables (there are L_k variables associated to the same waterlevel μ_k), the $a_{k,i}$'s and the $b_{k,i}$'s are given constant values, and the f_m 's are arbitrary functions that constrain the waterlevels (note that constraints on the $x_{k,i}$'s can be readily expressed as constraints on the μ_k 's simply by using the relation in (3)). We denote by $L_T = \sum_{k=1}^N L_k$ the total number of subchannels.

The numerical evaluation of the general waterfilling solution (3)-(4) may be of a high complexity. In order to make the problem more tractable, we consider a less general waterfilling solution by introducing an additional structure in the constraints. To be more precise, we consider $M = N + 1$ constraints of the form

$$\begin{aligned} f_k(\mu_k) &= t & 1 \leq k \leq N \\ g(\{\mu_k\}) &= 0 \end{aligned} \quad (5)$$

where the f_k 's are monotonic functions that constrain each waterlevel independently, g is a monotonic function that constrains all the waterlevels jointly, and t is an additional auxiliary variable introduced for generality. It is important to remark that a waterfilling solution with constraints as in (5) is still very general and, in fact, includes all the waterfilling solutions previously considered in the literature (to the authors' knowledge). Note that it is not difficult to generalize the constraints considered in (5), at the expense of a more involved notation, to include: more equality constraints at each k , inequality constraints at each k , a non-monotonic global constraint g , more global equality constraints, and global inequality constraints. (The assumption of the f_k 's being monotonic functions is necessary to reduce the complexity of the algorithm and make it practical.)

3 Practical Algorithms for Simple Waterfilling Solutions

For the simple case of single-waterlevel single-constrained waterfilling solutions, it is straightforward to obtain both iterative and exact algorithms. Iterative algorithms approach the exact solution as the number of iterations goes to infinity (in practice, of course, it suffices to stop iterating when the error is below a certain tolerance threshold). Exact algorithms give the exact result in a finite (and small) number of loops or iterations (with a worst-case complexity linear in the number of subchannels). The algorithms are based on the obvious fact that the solution is readily obtained once the waterlevel is known; therefore, the problem reduces to obtaining the waterlevel such that the constraint (typically, a power constraint) is satisfied.

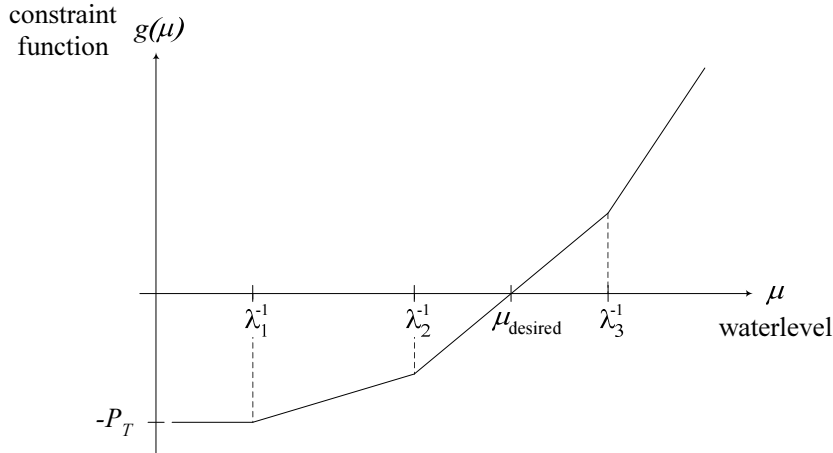


Figure 3: Illustration of the waterlevel determination as a problem of finding the root of the constraint function $g(\mu) = \sum_i x_i(\mu) - P_T$, where $x_i = (\mu - \lambda_i^{-1})^+$ and the λ_i 's are in decreasing order.

Iterative algorithms can be straightforwardly obtained simply by fixing the waterlevel μ to some value and then adjusting it iteratively until the constraint is satisfied [19, 10, 16] (in [13], a parametric approach was proposed to obtain the curve of power vs. performance, which also reduces to an iterative method when having to meet a specific power constraint). This is nothing else than the well-known problem of finding the root of a function (see Figure 3) which happens to be nonlinear and nondifferentiable (a power constraint, for example, would correspond to the constraint function $g(\mu) = \sum_i x_i(\mu) - P_T$, where the dependence of the x_i 's on the waterlevel μ has been made explicit). This can be done, for example, by modifying the waterlevel with small and (possibly) decreasing steps [10, 16] or by bisection, until the error is below some tolerance threshold.

Exact algorithms can also be easily derived based on hypothesis testing. The underlying idea is to form a hypothesis of the active ($x_i > 0$) and inactive ($x_i = 0$) subchannels and check whether a consistent solution can be found conditioned on the hypothesis. In principle, such an approach has an exponential worst-case complexity of 2^L , where L is the number of subchannels; a careful inspection of the structure of the problem, however, reveals that it is possible to obtain a linear worst-case complexity of L . In [20, Ch. 4] (see also [21, Ch. 7]) and [5, Alg. 1], an exact algorithm was obtained for the classical capacity-achieving solution. In [11], a similar algorithm was also derived for the minimization of the weighted sum of the MSEs of the subchannels. In a first attempt to unify the waterfilling solutions and their corresponding algorithms, practical algorithms were proposed in [22] for general waterfilling solutions with a single waterlevel and a power constraint (single constraint).

The following section is devoted to obtaining an exact unified algorithm for more general multi-waterlevel multi-constrained waterfilling solutions.

4 Unified Practical Algorithm for a Family of Waterfilling Solutions

This section provides the main result of the paper: an exact algorithm for the numerical evaluation of a general waterfilling solution with multiple waterlevels and multiple constraints as defined in (3) and (5). The interest of the following general algorithm is twofold: not only it is able to evaluate numerically a wide family of waterfilling solutions, but it is also an exact algorithm as opposed to iterative methods that approach the desired solution as the number of iterations goes to infinity (c.f. Section 3).

Proposition 1 *Suppose that a constrained optimization problem has a waterfilling solution of the form*

$$x_{k,i} = (\mu_k a_{k,i} - b_{k,i})^+ \quad 1 \leq k \leq N, 1 \leq i \leq L_k \quad (6)$$

subject to the constraints

$$f_k(\mu_k) = t \quad 1 \leq k \leq N \quad (7)$$

$$g(\{\mu_k\}) = 0 \quad (8)$$

where the μ_k 's are the N multiple waterlevels, the $x_{k,i}$'s are the variables (there are L_k variables associated to the same waterlevel μ_k), t is an auxiliary variable, the $a_{k,i}$'s and $b_{k,i}$'s are given positive¹ constant values, and the functions f_k 's and g are strictly increasing (the monotonicity can be non-strict in the range of μ_k 's corresponding to all $x_{k,i}$ equal to zero for some k). Then, the waterfilling solution can be numerically evaluated in practice with Algorithm 1 with a worst-case complexity of $L_T = \sum_{k=1}^N L_k$ iterations, i.e., a linear complexity in the total number of subchannels.

Algorithm 1 *Practical algorithm for the family of waterfilling solutions with multiple waterlevels and multiple constraints as described in Proposition 1.*

Input: Set of pairs $\{(a_{k,i}, b_{k,i})\}$ and constraint functions $\{f_k\}$ and g .

Output: Numerical solution $\{x_{k,i}\}$ and set of waterlevels $\{\mu_k\}$.

0. Set $\tilde{L}_k = L_k$ for $1 \leq k \leq N$ and (if necessary) sort the set of pairs $\{(a_{k,i}, b_{k,i})\}$ such that $\{a_{k,i}/b_{k,i}\}_{i=1}^{L_k}$ are in decreasing order $a_{k,i}/b_{k,i} \geq a_{k,i+1}/b_{k,i+1}$ for each k (define $a_{k,L_k+1}/b_{k,L_k+1} \triangleq 0$).

1. Compute

$$t^{\text{ub}} = \min_{1 \leq k \leq N} \left\{ f_k \left(b_{k, \tilde{L}_k+1} / a_{k, \tilde{L}_k+1} \right) \right\} \quad \text{and}$$

$$t^{\text{lb}} = \max_{1 \leq k \leq N} \left\{ f_k \left(b_{k, \tilde{L}_k} / a_{k, \tilde{L}_k} \right) \right\}$$

(denote the maximizing k by k_{max}).

¹The values $a_{k,i}$'s and $b_{k,i}$'s are assumed positive for simplicity of exposition. However, other cases can be similarly considered.

2. If $t^{\text{lb}} < t^{\text{ub}}$ and $g(\{\mu_k(t^{\text{lb}})\}) < 0$, where $\mu_k(t) = f_k^{-1}(t)$,² then accept the hypothesis and go to step 3.

Otherwise reject the hypothesis, form a new one by setting $\tilde{L}_{k_{\max}} = \tilde{L}_{k_{\max}} - 1$, and go to step 1.

3. Find $t_0 \in (t^{\text{lb}}, t^{\text{ub}}] \mid g(\{\mu_k(t_0)\}) = 0$, obtain the waterlevels and the numerical solution as

$$\begin{aligned}\mu_k &= f_k^{-1}(t_0), & 1 \leq k \leq N \\ x_{k,i} &= (\mu_k a_{k,i} - b_{k,i})^+, & 1 \leq k \leq N, 1 \leq i \leq L_k\end{aligned}$$

undo the sorting done at step 0 (if any), and finish.

Note that the algorithm requires the sequences $a_{k,i}$ and $b_{k,i}$ properly sorted. The complexity of the best sorting algorithm is $O(N \log_2 N)$ [25], where N denotes the length of the sequences; however, in many applications, such sequences are already sorted. This is because they come from the eigenvalues of a matrix and many of the algorithms to compute the eigenvalues and eigenvectors already produce the eigenvalues sorted [25, 26]. In any case, the complexity of sorting the eigenvalues is insignificant compared to the complexity $O(N^3)$ to compute the eigenvectors so that we can always assume the eigenvalues sorted.

Note also that, strictly speaking, we can only claim that the algorithm obtains the exact numerical solution when the equation $g(\{\mu_k(t_0)\}) = 0$ in the last step can be solved for t_0 in an exact way, e.g., when it can be solved in closed-form. Otherwise, some iterative method has to be used. In practice, however, this is not a problem since it is just a one-dimensional search which is done just once (after the algorithm has obtained the optimum hypothesis). Therefore, such cases are also contemplated (see, for example, Algorithm 3 in subsection 5.2.1).

Proof of Proposition 1. Algorithm 1 is based on hypothesis testing. First, it makes the hypothesis that all subchannels are active ($x_{k,i} > 0, \forall k, i$) and checks whether there is such a solution that satisfies all the constraints. If not, the hypothesis is rejected, a new hypothesis is made (properly chosen), and so forth. In principle, the total number of hypotheses is 2^{LT} (since each subchannel can be either active $x_{k,i} > 0$ or inactive $x_{k,i} = 0$), which implies an exponential worst-case complexity in the total number of subchannels. However, after a careful inspection of the problem structure, it turns out that the worst-case complexity can be made linear.

For each hypothesis, there is a set of possible values for the auxiliary variable t and, correspondingly, a set of possible values for the waterlevels μ_k 's. The problem is then to find the optimal hypothesis such that the waterlevels can be chosen to satisfy all the constraints. Interestingly, it is possible to obtain the exact way in which the hypotheses should be made such that the worst-case complexity is linear, starting from the hypothesis of all subchannels active and successively deactivating subchannels.

²As stated in Proposition 1, the functions f_k need not be strictly monotone for the range of μ_k in which all $x_{k,i}$ are zero for a given k . In such a case, we define the inverse as $f_k^{-1}(t) \triangleq \inf\{\mu_k : f_k(\mu_k) = t\}$ (this choice guarantees the minimum value of the function g).

See Appendix A for a detailed proof. ■

Since many optimization problems result in waterfilling solutions with a single waterlevel and a single constraint, it is interesting to particularize the previous result to this case.

Corollary 1 *Suppose that a constrained optimization problem has a waterfilling solution of the form*

$$x_i = (\mu a_i - b_i)^+ \quad 1 \leq i \leq L \quad (9)$$

subject to the constraint

$$g(\mu) = 0 \quad (10)$$

where μ is the waterlevel, the x_i 's are the L variables, the a_i 's and b_i 's are given positive constant values, and the function g is strictly increasing (the monotonicity can be nonstrict in the range of μ corresponding to all x_i equal to zero). Then, the waterfilling solution can be numerically evaluated in practice with Algorithm 2 with a worst-case complexity of L iterations, i.e., a linear complexity in the number of subchannels.³

Algorithm 2 *Practical algorithm for the family of single-level waterfilling solutions as described in Corollary 1.*

Input: *Set of pairs $\{(a_i, b_i)\}$ and constraint function g .*

Output: *Numerical solution $\{x_i\}$ and waterlevel μ .*

0. *Set $\tilde{L} = L$ and (if necessary) sort the set of pairs $\{(a_i, b_i)\}$ such that a_i/b_i are in decreasing order $a_i/b_i \geq a_{i+1}/b_{i+1}$ (define $a_{L+1}/b_{L+1} \triangleq 0$).*
1. *If $b_{\tilde{L}}/a_{\tilde{L}} < b_{\tilde{L}+1}/a_{\tilde{L}+1}$ and $g(b_{\tilde{L}}/a_{\tilde{L}}) < 0$, then accept the hypothesis and go to step 2. Otherwise reject the hypothesis, form a new one by setting $\tilde{L} = \tilde{L} - 1$, and go to step 1.*
2. *Find the waterlevel $\mu \in (b_{\tilde{L}}/a_{\tilde{L}}, b_{\tilde{L}+1}/a_{\tilde{L}+1}] \mid g(\mu) = 0$, obtain the numerical solution as*

$$x_i = (\mu a_i - b_i)^+ \quad 1 \leq i \leq L,$$

undo the sorting done at step 0 (if any), and finish.

Proof of Corollary 1. It follows directly from Proposition 1 and Algorithm 1 by introducing the innocuous constraint given by the function $f(\mu) = \mu$, from which we obtain $t^{\text{lb}} = b_{\tilde{L}}/a_{\tilde{L}}$ and $t^{\text{ub}} = b_{\tilde{L}+1}/a_{\tilde{L}+1}$. Note that Algorithm 2 reduces to that obtained in [22] for the particular case of having a power constraint $g(\mu) = \sum_i x_i - P_T$. ■

Summarizing, we can say that for any constrained optimization problem with a solution that fits into the general waterfilling structure as defined in Proposition 1 (or Corollary 1), it is not necessary to struggle to obtain heuristic and iterative methods to compute the solution numerically in practice. It suffices to particularize Algorithm 1 (or Algorithm 2) with the guaranty that it will give the exact numerical solution with a linear worst-case complexity in the number of subchannels.

³It is also possible to find the correct hypothesis following a bisection approach with a complexity of $\log_2(L)$.

5 Particularization to Different Examples of Waterfilling Solutions

In this section, we show how the general algorithms obtained in Section 4 for a wide family of waterfilling solutions can be readily particularized to several cases of interest in the design of communication systems.

5.1 Waterfilling Solutions with a Single Waterlevel

To start with, we consider some examples of simple waterfilling solutions with a single waterlevel and a single constraint and show how Algorithm 2 can be easily particularized.

5.1.1 Maximization of the mutual information subject to a power constraint

In communication systems, the maximization of the mutual information is a problem of a great interest since it gives the solution that achieves the channel capacity [1, 2, 3, 4, 5]. Such a capacity-achieving solution has a waterfilling form given by

$$x_i = (\mu - \lambda_i^{-1})^+ \quad 1 \leq i \leq L \quad (11)$$

subject to the power constraint

$$\sum_{i=1}^L x_i = P_T \quad (12)$$

where x_i is the power allocated to the i th subchannel, with gain λ_i , and P_T is the power available for transmission.

To obtain a practical algorithm, it suffices to use Algorithm 2 (single waterlevel and single constraint) introducing the following particularizations:

$$a_i = 1, \quad b_i = \lambda_i^{-1}$$

$$g(\mu) = \mu \tilde{L} - \sum_{i=1}^{\tilde{L}} \lambda_i^{-1} - P_T.$$

The comparison $g(b_{\tilde{L}}/a_{\tilde{L}}) < 0$ reduces to $\lambda_{\tilde{L}}^{-1} < (P_T + \sum_{i=1}^{\tilde{L}} \lambda_i^{-1}) / \tilde{L}$ and $\mu \mid g(\mu) = 0$ is obtained as $\mu = (P_T + \sum_{i=1}^{\tilde{L}} \lambda_i^{-1}) / \tilde{L}$.

5.1.2 Minimization of the sum of the MSEs subject to a power constraint

When the transmitter and receiver of a MIMO communication system are jointly designed to minimize the sum of the MSEs of the different subchannels (equivalently, the trace of the MSE matrix), the following waterfilling solution arises in the determination of the power distribution over the subchannels at the transmitter [8, 19, 9, 10, 6, 11, 7]:

$$x_i = \left(\mu \lambda_i^{-1/2} - \lambda_i^{-1} \right)^+ \quad 1 \leq i \leq L \quad (13)$$

subject to the power constraint

$$\sum_{i=1}^L x_i = P_T \quad (14)$$

where x_i is the power allocated to the i th subchannel, with gain λ_i , and P_T is the power available for transmission.

To obtain a practical algorithm, it suffices to use Algorithm 2 (single waterlevel and single constraint) introducing the following particularizations:

$$\begin{aligned} a_i &= \lambda_i^{-1/2}, \quad b_i = \lambda_i^{-1} \\ g(\mu) &= \mu \sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2} - \sum_{i=1}^{\tilde{L}} \lambda_i^{-1} - P_T. \end{aligned}$$

The comparison $g(b_{\tilde{L}}/a_{\tilde{L}}) < 0$ reduces to $\lambda_{\tilde{L}}^{-1/2} < (P_T + \sum_{i=1}^{\tilde{L}} \lambda_i^{-1}) / \sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2}$ and $\mu \mid g(\mu) = 0$ is obtained as $\mu = (P_T + \sum_{i=1}^{\tilde{L}} \lambda_i^{-1}) / \sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2}$.

5.1.3 Minimization of the transmit power subject to equal MSE constraints

Similarly to subsection 5.1.2, the joint design of the transmitter and receiver of a MIMO communication system to minimize the transmit power subject to equal MSE constraints for the different subchannels yields a waterfilling power allocation over the subchannels at the transmitter [23] given by

$$x_i = \left(\mu \lambda_i^{-1/2} - \lambda_i^{-1} \right)^+ \quad 1 \leq i \leq L \quad (15)$$

subject to the constraint

$$\sum_{i=1}^L \frac{1}{1 + x_i \lambda_i} = \rho \quad (16)$$

where x_i is the power allocated to the i th subchannel, with gain λ_i , and ρ/L is the MSE required at each subchannel.

To obtain a practical algorithm, it suffices to use Algorithm 2 (single waterlevel and single constraint) introducing the following particularizations:

$$\begin{aligned} a_i &= \lambda_i^{-1/2}, \quad b_i = \lambda_i^{-1} \\ g(\mu) &= - \left((L - \tilde{L}) + \mu^{-1} \sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2} - \rho \right). \end{aligned}$$

Note the additional sign in the definition of function g to make it increasing with respect to μ as required in Corollary 1. The comparison $g(b_{\tilde{L}}/a_{\tilde{L}}) < 0$ reduces to $\lambda_{\tilde{L}}^{-1/2} < (\sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2}) / (\rho - (L - \tilde{L}))$ and $\mu \mid g(\mu) = 0$ is obtained as $\mu = (\sum_{i=1}^{\tilde{L}} \lambda_i^{-1/2}) / (\rho - (L - \tilde{L}))$.

5.1.4 Minimization of the average BER subject to a power constraint

Since the BER is the ultimate performance measure of digital communications (rather than the MSE or SINR), the minimization of the BER is perhaps the most reasonable criterion to design a communication system. In [27], an OFDM system was optimized to minimize the BER averaged over the carriers; to be more precise, the waterfilling solution $x_i = \left(\mu \lambda_i^{1/(m+1)} - 1\right)^+ m/\lambda_i$ was obtained for the case with only statistic channel knowledge (following a Nakagami- m pdf) and $x_i = (\mu + \log \lambda_i)^+ / \lambda_i$ for perfect channel knowledge. Similarly, in [16], a parallel multiantenna MIMO system with minimum average BER was also considered with perfect channel knowledge and with additional constraints to guarantee a minimum BER per subchannel, obtaining a waterfilling solution with multiple waterlevels and multiple constraints which is briefly considered in subsection 5.2.3. However, after approximating the BER function with the Chernoff upper bound, the solution reduces to the following simple waterfilling solution which we now further develop for illustration purposes (after an appropriate change of notation with respect to [16, eq. (28)]):

$$x_i = (\mu - (c_i - \log \lambda_i))^+ \quad 1 \leq i \leq L \quad (17)$$

subject to

$$\sum_{i=1}^L (x_i + c_i) / \lambda_i = P_T \quad (18)$$

where the term $(x_i + c_i) / \lambda_i$ denotes the power allocated to the i th subchannel, with gain λ_i , c_i is the the minimum SINR required on the i th subchannel (to guarantee a minimum BER), and P_T is the power available for transmission.

To obtain a practical algorithm, it suffices to use Algorithm 2 (single waterlevel and single constraint) introducing the following particularizations:⁴

$$a_i = 1, \quad b_i = c_i - \log \lambda_i$$

$$g(\mu) = \sum_{i=1}^{\tilde{L}} (\mu + \log \lambda_i) / \lambda_i + \sum_{i=1+\tilde{L}}^L c_i / \lambda_i - P_T.$$

5.2 Waterfilling Solutions with Multiple Waterlevels

We now turn into more complicated waterfilling solutions with multiple waterlevels and multiple constraints and show how Algorithm 1 can be easily particularized.

5.2.1 Minimization of the maximum of the MSEs subject to a power constraint

The joint design of the transmitter and receiver to minimize the maximum of the MSEs of a communication through a multicarrier multi-antenna MIMO channel has been shown to be an excellent criterion

⁴To be exact, we have to add a constant value to each term $c_i - \log \lambda_i$ to make sure that the b_i 's are positive values. Such an approach, however, simplifies to the given expressions.

in terms of global performance since it guarantees the performance of the worst of the subchannels [7]. The corresponding waterfilling solution is

$$x_{k,i} = \left(\mu_k \lambda_{k,i}^{-1/2} - \lambda_{k,i}^{-1} \right)^+ \quad 1 \leq k \leq N, \quad 1 \leq i \leq L_k \quad (19)$$

subject to the constraints

$$\frac{1}{\check{L}_k} \left((\check{L}_k - L_k) + \sum_{i=1}^{L_k} \frac{1}{1 + x_{k,i} \lambda_{k,i}} \right) = t \quad 1 \leq k \leq N \quad (20)$$

$$\sum_{k=1}^N \sum_{i=1}^{L_k} x_{k,i} = P_T \quad (21)$$

where k and i denote the carrier and the spatial eigenmode respectively, N is the number of carriers, L_k is the number of used spatial eigenmodes at the k th carrier, $x_{k,i}$ is the power allocated to the (k, i) th subchannel, with gain $\lambda_{k,i}$, the \check{L}_k 's are given values, t is an auxiliary variable to be determined, and P_T is the power available for transmission (see [7] for details).

Since the waterfilling solution has multiple waterlevels and multiple constraints, Algorithm 1 can be used to obtain a practical algorithm with the following particularizations:

$$\begin{aligned} a_{k,i} &= \lambda_{k,i}^{-1/2}, \quad b_{k,i} = \lambda_{k,i}^{-1} \\ f_k(\mu_k) &= -\frac{1}{\check{L}_k} \left((\check{L}_k - \tilde{L}_k) + \mu_k^{-1} \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \\ g(\{\mu_k\}) &= \sum_{k=1}^N \mu_k \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} - \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - P_T. \end{aligned}$$

Note the additional sign in the definition of the f_k 's to make them increasing with respect to the μ_k 's as required in Proposition 1. From $f_k(\mu_k) = t$, it follows that

$$\mu_k = \frac{\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2}}{-t \check{L}_k - (\check{L}_k - \tilde{L}_k)}$$

and then

$$g(t) = \sum_{k=1}^N \frac{\left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2}{-t \check{L}_k - (\check{L}_k - \tilde{L}_k)} - \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - P_T.$$

The general algorithm reduces then to the following one (in [24], a similar algorithm was derived for this specific problem).

Algorithm 3 *Practical algorithm for the minimization of the maximum of the MSEs subject to a power constraint.*

Input: Set of subchannel gains $\{\lambda_{k,i}\}$ and maximum power P_T .

Output: Numerical solution $\{x_{k,i}\}$ and set of waterlevels $\{\mu_k\}$.

0. Set $\tilde{L}_k = L_k$ for $1 \leq k \leq N$ and (if necessary) sort the set of gains $\{\lambda_{k,i}\}$ such that $\{\lambda_{k,i}\}_{i=1}^{L_k}$ are in decreasing order $\lambda_{k,i} \geq \lambda_{k,i+1}$ for each k (define $\lambda_{k,L_k+1} \triangleq 0$).

1. Compute

$$t^{\text{ub}} = - \max_{1 \leq k \leq N} \left\{ \frac{1}{\tilde{L}_k} \left((\check{L}_k - \tilde{L}_k) + \lambda_{k,\tilde{L}_k+1}^{1/2} \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \right\} \quad \text{and}$$

$$t^{\text{lb}} = - \min_{1 \leq k \leq N} \left\{ \frac{1}{\tilde{L}_k} \left((\check{L}_k - \tilde{L}_k) + \lambda_{k,\tilde{L}_k}^{1/2} \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \right\}$$

(denote the minimizing k by k_{\min}).

2. If $t^{\text{lb}} < t^{\text{ub}}$ and $\sum_{k=1}^N \left(\left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2 / \left(-t^{\text{lb}} \check{L}_k - (\check{L}_k - \tilde{L}_k) \right) \right) < P_T + \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1}$, then accept the hypothesis and go to step 3.

Otherwise reject the hypothesis, form a new one by setting $\tilde{L}_{k_{\min}} = \tilde{L}_{k_{\min}} - 1$, and go to step 1.

3. Obtain t_0 , the waterlevels, and the numerical solution as

$$t_0 \in \left(t^{\text{lb}}, t^{\text{ub}} \right] : \sum_{k=1}^N \frac{\left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2}{-t_0 \check{L}_k - (\check{L}_k - \tilde{L}_k)} = P_T + \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1}$$

$$\mu_k = \frac{\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2}}{-t_0 \check{L}_k - (\check{L}_k - \tilde{L}_k)} \quad 1 \leq k \leq N, \quad \text{and}$$

$$x_{k,i} = \left(\mu_k \lambda_{k,i}^{-1/2} - \lambda_{k,i}^{-1} \right)^+ \quad 1 \leq k \leq N, \quad 1 \leq i \leq L_k,$$

undo the sorting done at step 0 (if any), and finish.

5.2.2 Maximization of the harmonic mean of the SINRs subject to a power constraint

Similarly to subsection 5.2.1, if the transmitter and receiver are jointly designed to maximize the harmonic mean of the SINRs of a communication through a MIMO channel, the following waterfilling solution is obtained [7]:

$$x_{k,i} = \left(\mu_k \lambda_{k,i}^{-1/2} - \lambda_{k,i}^{-1} \right)^+ \quad 1 \leq k \leq N, \quad 1 \leq i \leq L_k \quad (22)$$

subject to the constraints

$$\frac{\mu_k}{\check{L}_k^{1/2}} \left(L_k - \sum_{i=1}^{L_k} \frac{1}{1 + x_{k,i} \lambda_{k,i}} \right) = t \quad 1 \leq k \leq N \quad (23)$$

$$\sum_{k=1}^N \sum_{i=1}^{L_k} x_{k,i} = P_T \quad (24)$$

where the notation is as in (19)-(21).

The waterfilling solution has multiple waterlevels and multiple constraints and, therefore, Algorithm 1 can be used to obtain a practical algorithm with the following particularizations:

$$\begin{aligned}
a_{k,i} &= \lambda_{k,i}^{-1/2}, \quad b_{k,i} = \lambda_{k,i}^{-1} \\
f_k(\mu_k) &= \frac{1}{\tilde{L}_k^{1/2}} \left(\mu_k \tilde{L}_k - \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \\
g(\{\mu_k\}) &= \sum_{k=1}^N \mu_k \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} - \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - P_T.
\end{aligned}$$

From $f_k(\mu_k) = t$, it follows that

$$\mu_k = \frac{1}{\tilde{L}_k} \left(t \tilde{L}_k^{1/2} + \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)$$

and then

$$g(t) = t \sum_{k=1}^N \frac{\tilde{L}_k^{1/2}}{\tilde{L}_k} \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} + \sum_{k=1}^N \frac{1}{\tilde{L}_k} \left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2 - \sum_{k=1}^N \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - P_T.$$

The general algorithm reduces then to the following one (in [24], a similar algorithm was derived for this specific problem).

Algorithm 4 *Practical algorithm for the maximization of the harmonic mean of the SINRs subject to a power constraint.*

Input: Set of subchannel gains $\{\lambda_{k,i}\}$ and maximum power P_T .

Output: Numerical solution $\{x_{k,i}\}$ and set of waterlevels $\{\mu_k\}$.

0. Set $\tilde{L}_k = L_k$ for $1 \leq k \leq N$ and (if necessary) sort the set of gains $\{\lambda_{k,i}\}$ such that $\{\lambda_{k,i}\}_{i=1}^{\tilde{L}_k}$ are in decreasing order $\lambda_{k,i} \geq \lambda_{k,i+1}$ for each k (define $\lambda_{k,L_k+1} \triangleq 0$).

1. Compute

$$\begin{aligned}
t^{\text{ub}} &= \min_{1 \leq k \leq N} \left\{ \frac{1}{\tilde{L}_k^{1/2}} \left(\lambda_{k,\tilde{L}_k+1}^{-1/2} \tilde{L}_k - \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \right\} \quad \text{and} \\
t^{\text{lb}} &= \max_{1 \leq k \leq N} \left\{ \frac{1}{\tilde{L}_k^{1/2}} \left(\lambda_{k,\tilde{L}_k}^{-1/2} \tilde{L}_k - \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \right\}
\end{aligned}$$

(denote the maximizing k by k_{max}).

2. If $t^{\text{lb}} < t^{\text{ub}}$ and $t^{\text{lb}} < \left(P_T + \sum_{k=1}^N \left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - \left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2 / \tilde{L}_k \right) \right) / \left(\sum_{k=1}^N \tilde{L}_k^{1/2} / \tilde{L}_k \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)$, then accept the hypothesis and go to step 3.

Otherwise reject the hypothesis, form a new one by setting $\tilde{L}_{k_{\text{max}}} = \tilde{L}_{k_{\text{max}}} - 1$, and go to step 1.

3. Obtain t_0 , the waterlevels, and the numerical solution as

$$t_0 = \frac{P_T + \sum_{k=1}^N \left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1} - \frac{1}{\tilde{L}_k} \left(\sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right)^2 \right)}{\sum_{k=1}^N \frac{\tilde{L}_k^{1/2}}{\tilde{L}_k} \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2}}$$

$$\mu_k = \frac{1}{\tilde{L}_k} \left(t_0 \tilde{L}_k^{1/2} + \sum_{i=1}^{\tilde{L}_k} \lambda_{k,i}^{-1/2} \right) \quad 1 \leq k \leq N, \quad \text{and}$$

$$x_{k,i} = \left(\mu_k \lambda_{k,i}^{-1/2} - \lambda_{k,i}^{-1} \right)^+ \quad 1 \leq k \leq N, \quad 1 \leq i \leq \tilde{L}_k,$$

undo the sorting done at step 0 (if any), and finish.

5.2.3 Minimization of the average BER subject to a power constraint

As previously mentioned in subsection 5.1.4, the minimization of the BER averaged over the parallel subchannels existing within a MIMO channel results in a waterfilling solution with multiple waterlevels and multiple constraints [16] given by (after an appropriate change of notation with respect to [16, eqs. (25) and (26)])

$$x_k = \left(\mu_k - \frac{c_k}{\lambda_k} \right)^+ \quad 1 \leq k \leq N \quad (25)$$

subject to the constraints

$$\sqrt{\frac{\lambda_k}{\mu_k}} \exp(-\mu_k \lambda_k) = t \quad 1 \leq k \leq N \quad (26)$$

$$\sum_{k=1}^N \left(x_k + \frac{c_k}{\lambda_k} \right) = P_T \quad (27)$$

where the notation is as in (17)-(18) and t is an auxiliary variable to be determined.

This waterfilling solution, although including multiple waterlevels, is still simple to solve in practice since each waterlevel μ_k has a single associated variable x_k , *i.e.*, $L_k = 1$. As proposed in [16], the numerical computation is straightforward simply by iteratively updating the auxiliary variable t (from which the waterlevels μ_k 's and then the solution x_k can be readily obtained) until the power constraint is satisfied (c.f. Section 3). Alternatively, we can use the general result of Section 4, *i.e.*, Proposition 1 and Algorithm 1, to first obtain the optimum hypothesis and then the auxiliary variable $t_0 \in (t^{\text{lb}}, t^{\text{ub}}] \mid g(\{\mu_k(t_0)\}) = 0$ using an iterative approach as described above. Therefore, for this particular waterfilling solution, the use of Algorithm 1 (properly particularized) is useful only to obtain the optimum hypothesis and a small initial interval $(t^{\text{lb}}, t^{\text{ub}}]$ known to contain the optimum t to start the iterative search.

6 Conclusions

Many different constrained optimization problems result in waterfilling solutions for which some algorithms have been proposed in the literature to compute the numerical solution in practice.

This paper, instead of dealing with each specific waterfilling result, has considered a general waterfilling formulation with multiple waterlevels and multiple constraints, which embraces a wide family of cases, and has obtained an algorithm that provides the numerical solution with a linear worst-case complexity in the number of subchannels. In order to obtain an algorithm to evaluate a specific waterfilling solution, it suffices to particularize the general algorithm obtained in this paper to the problem at hand, without having to prove that such an algorithm gives indeed the desired solution with a linear worst-case complexity. In this sense, this paper bridges the gap between a wide family of waterfilling solutions and their efficient practical implementation.

7 Acknowledgment

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Appendix

A Proof of Proposition 1

As previously mentioned in the overview of the proof, the total number of hypotheses is, in principle, 2^{L_T} . However, as shown next, the worst-case complexity can be made linear. The first step in this simplification is obtained by sorting the set of pairs $\{(a_{k,i}, b_{k,i})\}_{i=1}^{L_k}$ such that $\{a_{k,i}/b_{k,i}\}_{i=1}^{L_k}$ are in decreasing order for each k (i.e., $a_{k,i}/b_{k,i} \geq a_{k,i+1}/b_{k,i+1}$). With this ordering and since $x_{k,i}/b_{k,i} = (\mu_k a_{k,i}/b_{k,i} - 1)^+$, a hypothesis is completely described by the set of active subchannels $\{\tilde{L}_k\}$ (such that $x_{k,i} > 0$ for $1 \leq k \leq N$, $1 \leq i \leq \tilde{L}_k$ and zero otherwise). The total number of hypotheses has been reduced to $\prod_{k=1}^N (L_k + 1)$ (since there are only $L_k + 1$ possibilities for each k), which can be seen to be smaller than the initial exponential complexity $2^{\sum_k L_k}$ by rewriting it as $2^{\sum_k \log_2(L_k + 1)}$ and noting that $\log_2(L_k + 1) \leq L_k$. In the following, this complexity will be further reduced to L_T by using the fact that the hypotheses of the different k 's are coupled.

First of all, let us analyze in detail the waterlevels μ_k 's. Consider a hypothesis given by $\{\tilde{L}_k\}$. The waterlevels for such a hypothesis must be such that the considered \tilde{L}_k subchannels are indeed active while the rest remain inactive:

$$\begin{aligned} \mu_k a_{k,i} - b_{k,i} \begin{cases} > 0 & 1 \leq k \leq N, 1 \leq i \leq \tilde{L}_k \\ \leq 0 & 1 \leq k \leq N, \tilde{L}_k < i \leq L_k \end{cases} \\ \iff \mu_k^{\text{lb}}(\tilde{L}_k) < \mu_k \leq \mu_k^{\text{ub}}(\tilde{L}_k) \quad 1 \leq k \leq N \end{aligned} \quad (28)$$

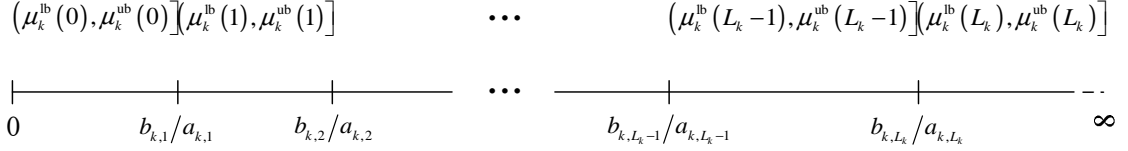


Figure 4: Illustration of the partition of the allowable waterlevels as a function of the hypothesis for a given index k .

where $\mu_k^{\text{lb}}(\tilde{L}_k) = b_{k,\tilde{L}_k}/a_{k,\tilde{L}_k}$ and $\mu_k^{\text{ub}}(\tilde{L}_k) = b_{k,\tilde{L}_k+1}/a_{k,\tilde{L}_k+1}$ (for convenience of notation, we define $\mu_k^{\text{lb}}(0) \triangleq 0$ and $\mu_k^{\text{ub}}(L_k) \triangleq \infty$). Therefore, simply because of the nature of the waterfilling solution, the waterlevels are lower- and upper-bounded for a given hypothesis. Noting that $\mu_k^{\text{ub}}(\tilde{L}_k) = \mu_k^{\text{lb}}(\tilde{L}_k + 1)$, it follows that the different hypotheses partition the set of waterlevels for a given k (see Figure 4):

$$\begin{aligned} \bigcup_{\tilde{L}_k=0}^{L_k} \left(\mu_k^{\text{lb}}(\tilde{L}_k), \mu_k^{\text{ub}}(\tilde{L}_k) \right] &= (0, \infty] \\ \left(\mu_k^{\text{lb}}(\tilde{L}_k^{(i)}), \mu_k^{\text{ub}}(\tilde{L}_k^{(i)}) \right] \cap \left(\mu_k^{\text{lb}}(\tilde{L}_k^{(j)}), \mu_k^{\text{ub}}(\tilde{L}_k^{(j)}) \right] &= \emptyset \quad \tilde{L}_k^{(i)} \neq \tilde{L}_k^{(j)}. \end{aligned} \quad (29)$$

Since functions f_k 's are strictly increasing, it follows from (28) that⁵

$$f_k \left(\mu_k^{\text{lb}}(\tilde{L}_k) \right) < f_k(\mu_k) \leq f_k \left(\mu_k^{\text{ub}}(\tilde{L}_k) \right) \quad 1 \leq k \leq N. \quad (30)$$

In addition, due to the constraints $f_k(\mu_k) = t$ for $1 \leq k \leq N$, the following bounds on t are obtained:

$$t_k^{\text{lb}}(\tilde{L}_k) < t \leq t_k^{\text{ub}}(\tilde{L}_k) \quad 1 \leq k \leq N \quad (31)$$

where $t_k^{\text{lb}}(\tilde{L}_k) = f_k \left(\mu_k^{\text{lb}}(\tilde{L}_k) \right)$ and $t_k^{\text{ub}}(\tilde{L}_k) = f_k \left(\mu_k^{\text{ub}}(\tilde{L}_k) \right)$. Note that the partition on the waterlevels of (29) (see Figure 4) translates into a partition on t for each k . More compactly, the previous set of bounds on t can be rewritten as

$$t^{\text{lb}} \left(\{ \tilde{L}_k \} \right) < t \leq t^{\text{ub}} \left(\{ \tilde{L}_k \} \right) \quad (32)$$

where $t^{\text{lb}} \left(\{ \tilde{L}_k \} \right) = \max_{1 \leq k \leq N} \left\{ t_k^{\text{lb}}(\tilde{L}_k) \right\}$ (denote the maximizing k by k_{max}) and $t^{\text{ub}} \left(\{ \tilde{L}_k \} \right) = \min_{1 \leq k \leq N} \left\{ t_k^{\text{ub}}(\tilde{L}_k) \right\}$.

For a given hypothesis, we have bounded the set of possible values for t and, correspondingly, for the waterlevels based on the form of the waterfilling solution $x_{k,i} = (\mu_k a_{k,i} - b_{k,i})^+$ and on the constraints $f_k(\mu_k) = t$. To finally decide whether the hypothesis is accepted or rejected, we still have to check whether there is a set of waterlevels $\{\mu_k\}$ within the bounded set such that $g(\{\mu_k\}) = 0$. It is convenient to rewrite g as a function of t (note that $\mu_k = f_k^{-1}(t)$) as

$$g(t) \triangleq g(\{\mu_k(t)\}) = g(\{f_k^{-1}(t)\}) \quad (33)$$

⁵We make an abuse of notation since the left inequality is nonstrict for $\tilde{L}_k = 0$, *i.e.*, $f_k(\mu_k^{\text{lb}}(0)) \leq f_k(\mu_k) \leq f_k(\mu_k^{\text{ub}}(0))$ for $1 \leq k \leq N$.

which is an increasing function since $g(\{\mu_k\})$ and each $\mu_k(t)$ are all increasing. The problem now is to check whether exists $t_0 \in \left(t^{\text{lb}}(\{\tilde{L}_k\}), t^{\text{ub}}(\{\tilde{L}_k\}) \right]$ such that $g(t_0) = 0$. Since $g(t)$ is increasing, it suffices to check whether $g\left(t^{\text{lb}}(\{\tilde{L}_k\})\right) < 0$ and $g\left(t^{\text{ub}}(\{\tilde{L}_k\})\right) \geq 0$.

We are now ready to derive a practical algorithm which starts with the largest possible value of t (largest values of the \tilde{L}_k 's) and then decreases it until all constraints are satisfied. In more detail, Algorithm 1 first reorders the subchannels as previously described to reduce the number of hypothesis to be checked and then sets $\tilde{L}_k = L_k$ for $1 \leq k \leq N$ to obtain the initial hypothesis. At this point, if all constraints can be satisfied, *i.e.*, if $g\left(t^{\text{lb}}(\{\tilde{L}_k\})\right) < 0$ and $g\left(t^{\text{ub}}(\{\tilde{L}_k\})\right) \geq 0$, the hypothesis is accepted. Otherwise, it is rejected and a new hypothesis has to be considered.

If a hypothesis is accepted, the optimal t_0 , waterlevels $\{\mu_k\}$, and solution $\{x_{k,i}\}$ are readily obtained as

$$t_0 \in \left(t^{\text{lb}}(\{\tilde{L}_k\}), t^{\text{ub}}(\{\tilde{L}_k\}) \right] \text{ such that } g(t_0) = 0 \quad (34)$$

$$\mu_k = f_k^{-1}(t_0) \quad 1 \leq k \leq N \quad (35)$$

$$x_{k,i} = (\mu_k a_{k,i} - b_{k,i})^+ \quad 1 \leq k \leq N, 1 \leq i \leq L_k. \quad (36)$$

If a hypothesis is rejected, a new one has to be made. We now show by induction how to do it in an efficient way such that the total number of hypotheses is reduced from $\prod_{k=1}^N (L_k + 1)$ to $\sum_{k=1}^N (L_k + 1)$. Assume that hypothesis $\{\tilde{L}_k\}$ has been rejected and that the optimal hypothesis (the one that contains the optimal t) is known not to contain larger values for any of the \tilde{L}_k 's (this clearly holds for the initial hypothesis since the values of the \tilde{L}_k 's are chosen as the highest possible). As a consequence, a new hypothesis can be made only by decreasing some of the \tilde{L}_k 's. Noting that if any of the \tilde{L}_k 's is decreased to some value \tilde{L}_k^{new} then $t_k^{\text{ub}}(\tilde{L}_k^{\text{new}}) \leq t_k^{\text{lb}}(\tilde{L}_k)$ and also that $t_k^{\text{lb}}(\tilde{L}_k) \leq t_{k_{\text{max}}}^{\text{lb}}(\tilde{L}_{k_{\text{max}}})$ (by definition of k_{max}), it follows that $t_k^{\text{ub}}(\tilde{L}_k^{\text{new}}) \leq t_{k_{\text{max}}}^{\text{lb}}(\tilde{L}_{k_{\text{max}}})$. This means that if a new hypothesis is made by decreasing some of the \tilde{L}_k 's for $k \neq k_{\text{max}}$, then the range of possible values for t will be empty since $\left(t_k^{\text{lb}}(\tilde{L}_k^{\text{new}}), t_k^{\text{ub}}(\tilde{L}_k^{\text{new}}) \right] \cap \left(t_{k_{\text{max}}}^{\text{lb}}(\tilde{L}_{k_{\text{max}}}), t_{k_{\text{max}}}^{\text{ub}}(\tilde{L}_{k_{\text{max}}}) \right] = \emptyset$. Thus, we can guarantee that the optimal hypothesis must have a lower value for $\tilde{L}_{k_{\text{max}}}$ of at most $\tilde{L}_{k_{\text{max}}} - 1$, which is taken as the next hypothesis to evaluate. Since it was assumed that the optimal hypothesis was known not to contain higher values for any of the \tilde{L}_k 's of the original hypothesis, the new hypothesis also satisfies this condition (since it only differs in $\tilde{L}_{k_{\text{max}}}$ and we have just shown that the optimal hypothesis has a value of $\tilde{L}_{k_{\text{max}}}$ of at most $\tilde{L}_{k_{\text{max}}} - 1$). Therefore, by induction (recall that the initial hypothesis also satisfies this condition), the previous mechanism to generate new hypotheses can be repeatedly applied.

By the nature of the algorithm, the maximum number of iterations (worst-case complexity) is $\sum_{k=1}^N (L_k + 1) = L_T + N \simeq L_T$.

As a final remark, when evaluating a given hypothesis, note that the condition $g\left(t^{\text{ub}}(\{\tilde{L}_k\})\right) \geq 0$ is always satisfied in the initial hypothesis and it is sufficient to check only the condition $g\left(t^{\text{lb}}(\{\tilde{L}_k\})\right) < 0$. In the subsequent hypotheses, it is again sufficient to check only this condi-

tion since the other condition $g\left(t^{\text{ub}}\left(\{\tilde{L}_k\}\right)\right) \geq 0$ is always satisfied as well (because it was implicitly evaluated in some previous rejected hypothesis). ■

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