

# Competitive Design of Multiuser MIMO Systems Based on Game Theory: A Unified View

Gesualdo Scutari, *Member, IEEE*, Daniel P. Palomar, *Member, IEEE*, and Sergio Barbarossa, *Member, IEEE*

**Abstract**—This paper considers the noncooperative maximization of mutual information in the Gaussian interference channel in a fully distributed fashion via game theory. This problem has been studied in a number of papers during the past decade for the case of frequency-selective channels. A variety of conditions guaranteeing the uniqueness of the Nash Equilibrium (NE) and convergence of many different distributed algorithms have been derived. In this paper we provide a unified view of the state-of-the-art results, showing that most of the techniques proposed in the literature to study the game, even though apparently different, can be unified using our recent interpretation of the waterfilling operator as a projection onto a proper polyhedral set. Based on this interpretation, we then provide a mathematical framework, useful to derive a unified set of sufficient conditions guaranteeing the uniqueness of the NE and the global convergence of waterfilling based asynchronous distributed algorithms.

The proposed mathematical framework is also instrumental to study the extension of the game to the more general MIMO case, for which only few results are available in the current literature. The resulting algorithm is, similarly to the frequency-selective case, an iterative asynchronous MIMO waterfilling algorithm. The proof of convergence hinges again on the interpretation of the MIMO waterfilling as a matrix projection, which is the natural generalization of our results obtained for the waterfilling mapping in the frequency-selective case.

**Index Terms**—Game Theory, MIMO Gaussian interference channel, Nash equilibrium, totally asynchronous algorithms, waterfilling.

## I. INTRODUCTION

THE interference channel is a mathematical model relevant to many physical communication channels and multiuser systems where multiple uncoordinated links share a common communication medium, such as digital subscriber lines [1], single (or multi) antenna cellular radio, ad-hoc wireless networks [2], [3], and cognitive radio systems [4].

The interference channel is characterized by its capacity region, defined as the set of rates that can be simultaneously achieved by the users in the system while making the error probability arbitrary small. A pragmatic approach that leads to an achievable region or inner bound of the capacity region is to restrict the system to operate as a set of independent units, i.e., not allowing multiuser encoding/decoding or the use of interference cancellation techniques. This approach is

Manuscript received August 15, 2007; revised May 10, 2008. This work was supported by the RPC07/08.EG23 project.

G. Scutari and S. Barbarossa are with the INFOCOM Department, University of Rome, “La Sapienza,” Rome, Italy (e-mails: {scutari, sergio}@infocom.uniroma1.it).

D. P. Palomar is with the Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong (e-mail: palomar@ust.hk).

Digital Object Identifier 10.1109/JSAC.2008.080907.

very relevant in practical systems, as it limits the amount of signaling among the users. With this assumption, multiuser interference is treated as additive colored noise and the system design reduces to finding the optimum covariance matrix of the symbols transmitted by each user.

Within this context, in this paper we consider the maximization of mutual information in a fully distributed fashion using a game theoretical approach. Since the seminal paper of Yu et al. [7] in 2002 (and the conference version in 2001), this problem has been studied in a number of works during the past seven years for the case of *SISO frequency-selective channels* or, equivalently, a set of parallel non-interfering scalar channels [8]–[20]. In the cited papers, the maximization of mutual information is formulated as a strategic noncooperative game, where every SISO link is a player that competes against the others by choosing his power allocation (transmission strategy) over the frequency bins (or parallel channels) to maximize his own information rate (payoff function).<sup>1</sup> Based on the celebrated notion of Nash Equilibrium (NE) in game theory (cf. [5], [6]), an equilibrium for the whole system is reached when every player’s reaction is “unilaterally optimal”, i.e., when, given the rival players’ current strategies, any change in a player’s own strategy would result in a rate loss. This vector-valued power control game was widely studied and several sufficient conditions have been derived that guarantee the uniqueness of the NE and the convergence of alternative distributed waterfilling based algorithms: synchronous sequential [7]–[14], synchronous simultaneous [14], [15], [16], [19], and asynchronous [17], [20].

Interestingly, different approaches have been used in the cited papers to analyze the game, most of them based on the following, apparently different, key results: 1) the interpretation of the waterfilling operator as a projection onto a proper polyhedral set [14], [19]; 2) the interpretation of the Nash equilibria of the game as solutions of a proper affine Variational Inequality (VI) problem [12]; and 3) the interpretation of the waterfilling mapping as a piecewise affine function [16], [40, Ch. 4]. In this paper, we provide a unified view of these results, showing that they fit naturally in our interpretation of the waterfilling mapping as a projector [14],

<sup>1</sup>The choice of the gender of the players is always controversial in the literature of game theory. This is reflected mainly on the use the third-person singular pronouns: some authors use “his”, while others use “her”, and others—more diplomatic ones—even use “his/her”. English non-native speakers tend to use “its” to avoid the problem but that is not well accepted by native speakers. In two-player zero-sum games, the issue is even trickier and some authors resort to the use of one gender for the good player and another for the bad player. See the foreword in [5] for a related discussion on the issue. In this paper, for simplicity of notation and without further implications, we simply use “his”.

[19]. Building on this interpretation and using classical results from fixed-point and contraction theory (cf. [22], [24], [25], [26]), we then develop a mathematical framework useful to derive a unified set of sufficient conditions guaranteeing both the uniqueness of the NE and the convergence of totally asynchronous iterative waterfilling based algorithms.

The proposed mathematical framework is instrumental to study the more general MIMO case, which is a nontrivial extension of the SISO frequency-selective case. There are indeed only a few papers that have studied (special cases of) the MIMO game [21], [27]-[31]. In [27], the authors focused on the two-user MISO channel. In [28]-[30], the authors considered the rate maximization game in MIMO interference channels, but they provided only numerical results to support the existence of a NE of the game. Furthermore, in these papers there is no study of the uniqueness of the equilibrium and convergence of the proposed algorithms. Finally, in [31], the authors showed that the MIMO rate maximization game is a concave game (in the sense of [32]), implying the existence of a NE for any set of arbitrary channel matrices [32, Theorem 1]. As far as the uniqueness of the equilibrium is concerned, they only showed that if the multiuser interference is almost negligible, then the NE is unique, without quantifying how small the interference must be. Hence, a practical condition that one can check to guarantee the uniqueness of the NE of the game and convergence of distributed algorithms is currently missing.

The main difficulty in the MIMO case is that the optimal transmit directions (i.e., eigenvectors of the transmit covariance matrix) of each user change with the strategies of the other users, as opposed to the SISO frequency-selective case where only the power allocation depends on the strategies of the others, but the directions remain fixed: i) in the diagonal MIMO case, the transmit directions are always the canonical vectors [7]-[14]; ii) in the frequency-selective channel, the transmit directions are the Fourier vectors [18], [19]; iii) for the MISO case, the transmit directions are matched to the vector channels; and iv) for the SIMO case, there are no transmit directions to optimize. For the previous reason, the existing results and techniques in [7]-[20], valid for SISO frequency-selective channels, cannot be applied or trivially extended to the MIMO case. On top of that, another difficulty is the fact that, differently from the vector power control game in [7]-[19], where there exists an explicit relationship (via the waterfilling solution) among the optimal power allocations of all the users, in the matrix-valued MIMO game one cannot obtain an explicit expression of the optimal covariance matrix of each user at the NE (the MIMO waterfilling solution), as a function of the optimal covariance matrices of the other users, but there exists only a complicated implicit relationship, via an eigendecomposition.

Building on the mathematical framework developed for the SISO case, we can overcome the main difficulties in the study of the MIMO game invoking a novel interpretation of the MIMO waterfilling operator as a projector and its nonexpansion property, similar to the one for frequency-selective channels. This enables us to derive a unified set of sufficient conditions that guarantee the uniqueness of the Nash equilibrium of the MIMO game and the convergence

of totally asynchronous distributed algorithms based on the MIMO waterfilling solution.

The paper is organized as follows. Section II gives the system model and formulates the optimization problem as a strategic noncooperative game. In Section III, we draw the relationship between Nash equilibria of the game and fixed points of nonlinear sets of equations, and provide the mathematical tools necessary to study convergence of distributed asynchronous algorithms. Building on the interpretation of the multiuser waterfilling solution as a proper projection onto a convex set, in Section IV, we provide the main properties of the multiuser waterfilling solution either in the SISO or MIMO case, unifying previous results proposed in the literature to study the rate maximization game in SISO frequency-selective channels. The contraction property of the multiuser waterfilling paves the way to derive sufficient conditions guaranteeing the uniqueness of the fixed point of the waterfilling projector—alias the NE of the (SISO/MIMO) game—and the convergence of iterative, possibly asynchronous, distributed algorithms, as detailed in Sections V and VI, respectively. Section VII reports some numerical results illustrating the benefits of MIMO transceivers in the multiuser context. Finally, Section VIII draws some conclusions.

## II. SYSTEM MODEL AND PROBLEM FORMULATION

In this section we introduce the system model and formulate the optimization problem addressed in the paper explicitly.

### A. System Model

We consider a vector Gaussian interference channel composed of  $Q$  links. In this model, there are  $Q$  transmitter-receiver pairs, where each transmitter wants to communicate with its corresponding receiver over a MIMO channel. The transmission over the generic  $q$ -th MIMO channel with  $n_{T_q}$  transmit and  $n_{R_q}$  receive dimensions can be described with the baseband signal model

$$\mathbf{y}_q = \mathbf{H}_{qq}\mathbf{x}_q + \sum_{r \neq q} \mathbf{H}_{rq}\mathbf{x}_r + \mathbf{n}_q, \quad (1)$$

where  $\mathbf{x}_q \in \mathbb{C}^{n_{T_q}}$  is the vector transmitted by source  $q$ ,  $\mathbf{H}_{qq} \in \mathbb{C}^{n_{R_q} \times n_{T_q}}$  is the direct channel of link  $q$ ,  $\mathbf{H}_{rq} \in \mathbb{C}^{n_{R_q} \times n_{T_r}}$  is the cross-channel matrix between source  $r$  and destination  $q$ ,  $\mathbf{y}_q \in \mathbb{C}^{n_{R_q}}$  is the vector received by destination  $q$ , and  $\mathbf{n}_q \in \mathbb{C}^{n_{R_q}}$  is a zero-mean circularly symmetric complex Gaussian noise vector with arbitrary covariance matrix  $\mathbf{R}_{n_q}$  (assumed to be nonsingular). The second term on the right-hand side of (1) represents the Multi-User Interference (MUI) received by the  $q$ -th destination and caused by the other active links. For each transmitter  $q$ , the total average transmit power is

$$\mathcal{E} \left\{ \|\mathbf{x}_q\|_2^2 \right\} = \text{Tr}(\mathbf{Q}_q) \leq P_q, \quad (2)$$

where  $\text{Tr}(\cdot)$  denotes the trace operator,  $\mathbf{Q}_q \triangleq \mathcal{E} \left\{ \mathbf{x}_q \mathbf{x}_q^H \right\}$  is the covariance matrix of the transmitted vector  $\mathbf{x}_q$ , and  $P_q$  is the maximum average transmitted power in units of energy per transmission.

The system model in (1)-(2) provides a unified way to represent many physical communication channels and multiuser

systems of practical interest. What changes from one system to the other is the structure of the channel matrices. We may have, in fact, as particular cases of (1)-(2): i) digital subscriber lines [1], where the channel matrices are Toeplitz circulant, the matrices  $\mathbf{Q}_q = \mathbf{W} \text{Diag}(\mathbf{p}_q) \mathbf{W}^H$  incorporate the DFT precoding  $\mathbf{W}^H$ , the vectors  $\mathbf{p}_q$  allocate the power across the frequency bins, and the MUI is mainly caused by near-end cross talk; ii) single (or multi) antenna CDMA cellular radio systems, where the matrices  $\mathbf{Q}_q = \mathbf{F}_q \mathbf{F}_q^H$  contain in  $\mathbf{F}_q$  the user codes within a given cell, and the MUI is essentially intercell interference [2]; iii) ad-hoc wireless MIMO networks, where the channel matrices represent the MIMO channel of each link [3].

Since our goal is to find distributed algorithms that do not require neither a centralized control nor a coordination among the links, we focus on transmission techniques where no interference cancellation is performed and multiuser interference is treated as additive colored noise from each receiver. Each channel is assumed to change sufficiently slowly to be considered fixed during the whole transmission, so that the information theoretical results are meaningful. Moreover, perfect channel state information at both transmitter and receiver sides of each link is assumed;<sup>2</sup> each receiver is also assumed to measure with no errors the covariance matrix of the noise plus MUI generated by the other users. Finally, we assume that the channel matrices  $\mathbf{H}_{qq}$  are square nonsingular. The more general case of possibly rectangular nonfull rank matrices is addressed in [21].

Under these assumptions, invoking the capacity expression for the single user Gaussian MIMO channel—achievable using random Gaussian codes by all the users—the maximum information rate on link  $q$  for a given set of users' covariance matrices  $\mathbf{Q}_1, \dots, \mathbf{Q}_Q$  is [33]

$$R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) = \log \det (\mathbf{I} + \mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \mathbf{H}_{qq} \mathbf{Q}_q) \quad (3)$$

where  $\mathbf{R}_{-q}(\mathbf{Q}_{-q}) \triangleq \mathbf{R}_{n_q} + \sum_{r \neq q} \mathbf{H}_{rq} \mathbf{Q}_r \mathbf{H}_{rq}^H$  is the MUI plus noise covariance matrix observed by user  $q$ , and  $\mathbf{Q}_{-q} \triangleq (\mathbf{Q}_r)_{r=1, r \neq q}^Q$  is the set of all the users' covariance matrices, except the  $q$ -th one.

## B. Game Theoretical Formulation

We formulate the system design within the framework of game theory using as desirable criterion the concept of Nash Equilibrium (NE) (cf. [5], [6]). Specifically, we consider a strategic noncooperative game, in which the players are the links and the payoff functions are the information rates on each link: Each player  $q$  competes against the others by choosing his transmit covariance matrix  $\mathbf{Q}_q$  (i.e., his strategy) that maximizes his own information rate  $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$  in (3), subject to the transmit power constraint (2). A solution of the game—a NE—is reached when each user, given the strategy profiles of the others, does not get any rate increase by unilaterally changing his own strategy. Stated in mathematical

terms, the game has the following structure:

$$(\mathcal{G}) : \begin{array}{ll} \underset{\mathbf{Q}_q}{\text{maximize}} & R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}) \\ \text{subject to} & \mathbf{Q}_q \in \mathcal{Q}_q, \end{array} \quad \forall q \in \Omega, \quad (4)$$

where  $\Omega \triangleq \{1, \dots, Q\}$  is the set of players (i.e., the links);  $R_q(\mathbf{Q}_q, \mathbf{Q}_{-q})$  defined in (3) is the payoff function of player  $q$ ; and  $\mathcal{Q}_q$  is the set of admissible strategies (the covariance matrices) for player  $q$ , defined as<sup>3</sup>

$$\mathcal{Q}_q \triangleq \{ \mathbf{Q} \in \mathbb{C}^{n_{T_q} \times n_{T_q}} : \mathbf{Q} \succeq \mathbf{0}, \text{Tr}\{\mathbf{Q}\} = P_q \}. \quad (5)$$

The solutions of game  $\mathcal{G}$  are formally defined as follows.<sup>4</sup>

*Definition 1:* A (pure) strategy profile  $\mathbf{Q}^* = (\mathbf{Q}_q^*)_{q \in \Omega} \in \mathcal{Q}_1 \times \dots \times \mathcal{Q}_Q$  is a NE of game  $\mathcal{G}$  if

$$R_q(\mathbf{Q}_q^*, \mathbf{Q}_{-q}^*) \geq R_q(\mathbf{Q}_q, \mathbf{Q}_{-q}^*), \quad \forall \mathbf{Q}_q \in \mathcal{Q}_q, \forall q \in \Omega. \quad (6)$$

□

To write the Nash equilibria of game  $\mathcal{G}$  in a convenient form, we first introduce the following notations and definitions. Given  $\mathcal{G}$ , for each  $q \in \Omega$  and  $\mathbf{Q}_{-q} \in \mathcal{Q}_{-q} \triangleq \mathcal{Q}_1 \times \dots \times \mathcal{Q}_{q-1} \times \mathcal{Q}_{q+1} \times \dots \times \mathcal{Q}_Q$ , we write the eigendecomposition of  $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \mathbf{H}_{qq}$  as:

$$\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}) \mathbf{H}_{qq} = \mathbf{U}_q \mathbf{D}_q \mathbf{U}_q^H, \quad (7)$$

where  $\mathbf{U}_q = \mathbf{U}_q(\mathbf{Q}_{-q}) \in \mathbb{C}^{n_{T_q} \times n_{T_q}}$  is a unitary matrix with the eigenvectors,  $\mathbf{D}_q = \mathbf{D}_q(\mathbf{Q}_{-q}) \in \mathbb{R}_{++}^{n_{T_q} \times n_{T_q}}$  is a diagonal matrix with the  $n_{T_q}$  positive eigenvalues, and  $\mathbf{R}_{-q}(\mathbf{Q}_{-q}) = \mathbf{R}_{n_q} + \sum_{r \neq q} \mathbf{H}_{rq} \mathbf{Q}_r \mathbf{H}_{rq}^H$ .

Given  $q \in \Omega$  and  $\mathbf{Q}_{-q} \in \mathcal{Q}_{-q}$ , the solution to problem (4) is the well-known waterfilling solution (e.g., [33]):

$$\mathbf{Q}_q^* = \mathbf{W}\mathbf{F}_q(\mathbf{Q}_{-q}), \quad (8)$$

with the waterfilling operator  $\mathbf{W}\mathbf{F}_q(\cdot)$  defined as

$$\mathbf{W}\mathbf{F}_q(\mathbf{Q}_{-q}) \triangleq \mathbf{U}_q (\mu_q \mathbf{I} - \mathbf{D}_q^{-1})^+ \mathbf{U}_q^H, \quad (9)$$

where  $\mathbf{U}_q = \mathbf{U}_q(\mathbf{Q}_{-q})$ ,  $\mathbf{D}_q = \mathbf{D}_q(\mathbf{Q}_{-q})$ , and  $\mu_q$  is chosen to satisfy  $\text{Tr}\{(\mu_q \mathbf{I} - \mathbf{D}_q^{-1})^+\} = P_q$ , with  $(x)^+ \triangleq \max(0, x)$ .

Using (8) and Definition 1, we can now characterize the Nash Equilibria of the game  $\mathcal{G}$  in a compact way as the following waterfilling fixed-point equation:

$$\mathbf{Q}_q^* = \mathbf{W}\mathbf{F}_q(\mathbf{Q}_{-q}^*), \quad \forall q \in \Omega. \quad (10)$$

*Remark 1 - Competitive maximization of transmission rates.* The choice of the objective function as in (3) requires the use of ideal Gaussian codebooks with a proper covariance matrix. In practice, Gaussian codes may be substituted with simple (suboptimal) finite-order signal constellations, such as

<sup>3</sup>Observe that, in the definition of  $\mathcal{Q}_q$  in (5) the condition  $\mathbf{Q} = \mathbf{Q}^H$  is redundant, since any complex positive semidefinite matrix must be necessarily Hermitian [41, Sec. 7.1]. Furthermore, we replaced, without loss of generality (w.l.o.g.), the original inequality power constraint in (2) with equality, since, at the optimum to each problem in (4), the constraint must be satisfied with equality.

<sup>4</sup>Observe that, for the payoff functions defined in (3), we can indeed limit ourselves to adopt pure strategies w.l.o.g., as we did in (3), since every NE of the game can be proved to be achievable using pure strategies [18].

<sup>2</sup>Note that each user  $q$  is only required to know his own channel  $\mathbf{H}_{qq}$ , but not the cross-channels  $\{\mathbf{H}_{rq}\}_{r \neq q}$  from the other users.

Quadrature Amplitude Modulation (QAM) or Pulse Amplitude Modulation (PAM), and practical (suboptimal) coding schemes. In this case, instead of considering the maximization of *mutual information* on each link, one can focus on the competitive maximization of the *transmission rate*, using finite order constellation, under constraints on transmit power and on the average error probability  $P_{e,q}^*$  (see [18] for more details). Interestingly, using a similar approach to that in [18], one can prove that the optimal transmission strategy of each user is still a solution to the fixed-point equation in (10), where each channel matrix  $\mathbf{H}_{qq}$  is replaced by  $\mathbf{H}_{qq}/\Gamma_q$ , where  $\Gamma_q \geq 1$  denotes the gap, which depends only on the constellations and on the target error probability  $P_{e,q}^*$  [34].

*Remark 2 - Related works.* The matrix nature of game  $\mathcal{G}$  and the arbitrary structure of the channel matrices make the analysis of the game quite complicated, since none of the results in game theory literature can be directly applied to characterize solutions of the form (10). The main difficulty in the analysis comes from the fact that, for each user  $q$ , the optimal eigenvector matrix  $\mathbf{U}_q^* = \mathbf{U}_q(\mathbf{Q}_{-q}^*)$  in (9) depends, in general, on the strategies  $\mathbf{Q}_{-q}^*$  of all the others, through a very complicated implicit relationship—the eigendecomposition of the equivalent channel matrix  $\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q}^*) \mathbf{H}_{qq}$ . In the vector power control games studied in [7]–[20], the analysis of uniqueness of the equilibrium was mathematically more tractable, since scalar frequency-selective channels are represented by diagonal matrices (or Toeplitz and circulant matrices [18], [19]), implying that the optimal set of eigenvectors of any NE becomes *user-independent* [18]. In the present case, it follows that, because of the dependence of the optimal strategy  $\mathbf{U}_q^*(\mathbf{Q}_{-q}^*)$  of each user on the strategy profile of the others at the NE, one cannot use the uniqueness condition of the NE obtained in [7]–[20] to guarantee the uniqueness of the NE of game  $\mathcal{G}$  in (4), even if game  $\mathcal{G}$  reduces to the power control game studied in the cited papers, once the optimal users’ strategy profile in (10) is introduced in (4). At the best of our knowledge, the only paper where game  $\mathcal{G}$  was partially analyzed is [31], where the authors applied the framework developed in [32] to the MIMO game and showed that the NE becomes unique if the MUI in the system—the interference-to-noise ratio at each receiver—is sufficiently small, but without quantifying exactly how much small the MUI must be. Thus, a practical condition that one can check to guarantee the uniqueness of the NE is still missing.

To overcome the difficulties in the study of game  $\mathcal{G}$ , we propose next an equivalent expression of the waterfilling solution enabling us to express the Nash equilibria in (10) as a fixed-point of a more tractable mapping. This alternative expression is based on the new interpretation of the MIMO waterfilling solution as a proper projector operator. Based on this result, we can then derive sufficient conditions for the uniqueness of the NE and convergence of asynchronous distributed algorithms, as detailed in Sections V and VI respectively.

### III. NASH EQUILIBRIUM AS A FIXED POINT

Before providing one of the major results of the paper—the contraction properties of the MIMO multiuser waterfilling projector—we recall and unify some standard results from

fixed-point [25] and contraction theory [22], [24] that will be instrumental for our derivations (recall from (10) that any NE can be interpreted as a fixed point of the waterfilling mapping). The proposed unified mathematical framework is also useful to establish an interesting link among the alternative, apparently different, approaches proposed in the literature to study the rate maximization game in SISO frequency-selective interference channels [7]–[14], showing that most of the results in [7]–[14] can be unified by our interpretation of the waterfilling as a projector [14], [19].

#### A. Existence and Uniqueness of a Fixed-point

Let  $\mathbf{T} : \mathcal{X} \mapsto \mathcal{X}$  be any mapping from a subset  $\mathcal{X} \subseteq \mathbb{R}^n$  to itself. One can associate  $\mathbf{T}$  to a dynamical system described by the following discrete-time equation:

$$\mathbf{x}(n+1) = \mathbf{T}(\mathbf{x}(n)), \quad n \in \mathbb{N}_+, \quad (11)$$

where  $\mathbf{x}(n) \in \mathbb{R}^n$  is the vector of the state variables of the system at (discrete) time  $n$ , with  $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ . The equilibria of the system, if they exist, are the vectors  $\mathbf{x}^*$  resulting as a solution of  $\mathbf{x}^* = \mathbf{T}(\mathbf{x}^*)$ , i.e., the fixed-points of mapping  $\mathbf{T}$ . The study of the existence and uniqueness of an equilibrium of a dynamical system has been widely addressed either in fixed-point theory (cf. [22], [24], [25]) or control theory (cf. [26], [35], [36]) literature. Many alternative conditions are available. Throughout this paper, we will use the following.

*Theorem 1:* Given the dynamical system in (11) with  $\mathbf{T} : \mathcal{X} \mapsto \mathcal{X}$  and  $\mathcal{X} \subseteq \mathbb{R}^n$ , we have the following:

- Existence (cf. [24], [25]): If  $\mathcal{X}$  is nonempty, convex and compact, and  $\mathbf{T}$  is a continuous mapping, then there exists some  $\mathbf{x}^*$  such that  $\mathbf{x}^* = \mathbf{T}(\mathbf{x}^*)$ ;
- Uniqueness (cf. [22], [24], [25]): If  $\mathcal{X}$  is closed and  $\mathbf{T}$  is a contraction in some vector norm  $\|\cdot\|$ , with modulus  $\alpha \in [0, 1)$ , i.e.,

$$\|\mathbf{T}(\mathbf{x}^{(1)}) - \mathbf{T}(\mathbf{x}^{(2)})\| \leq \alpha \|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\|, \quad \forall \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathcal{X}, \quad (12)$$

then the fixed-point of  $\mathbf{T}$  is unique.  $\square$

*Remark 3 - Sufficiency of the conditions.* The conditions of Theorem 1 are only sufficient for the existence and uniqueness of the fixed point. However, this does not mean that some of them can be removed. For example, the convexity assumption in the existence condition cannot, in general, be removed, as the simple one-dimensional example  $T(x) = -x$  and  $\mathcal{X} = \{-c, c\}$ , with  $c \in \mathbb{R}$ , shows.

*Remark 4 - Choice of the norm.* The contractive property of the mapping is *norm-dependent*, in the sense that a mapping may be contractive for some choice of the norm on  $\mathbb{R}^n$  and, at the same time, it may fail to be so under a different norm. On the other hand, it may happen that a mapping is a contraction in more than one norm. In such a case, even though the uniqueness of the fixed-point is guaranteed whatever the choice of the norm is (cf. Theorem 1), the convergence of different algorithms, based on the same mapping  $\mathbf{T}$ , to the fixed-point is, in general, norm-dependent. Thus, the choice of the proper norm is a critical issue and it actually gives us

potential degrees of freedom to be explored in the characterization of the convergence properties of the desired algorithms used to reach the fixed-point. We address this issue in the next sections, where we introduce a proper norm, tailored to our needs.

### B. Convergence to a Fixed-point

Nonlinear fixed-point problems are typically solved by iterative methods, especially when one is interested in distributed algorithms [22], [24]. In fact, the mapping  $\mathbf{x} \mapsto \mathbf{T}(\mathbf{x})$  can be interpreted as an algorithm for finding such a fixed point. The degrees of freedom are in the choice of the specific updating scheme among the components of vector  $\mathbf{x}$ , based on mapping  $\mathbf{T}$ . More specifically, denoting by  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_Q)$  a partition of  $\mathbf{x}$ , with  $\mathbf{x}_q \in \mathbb{R}^{n_q}$  and  $n_1 + \dots + n_Q = n$ , and assuming  $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_Q$ ,<sup>5</sup> with each  $\mathcal{X}_q \subseteq \mathbb{R}^{n_q}$ , the most common updating strategies for  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  based on mapping  $\mathbf{T}$  are [22], [24]:

- i) *Jacobi scheme*: All components  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  are updated *simultaneously*, via the mapping  $\mathbf{T}$ ;
- ii) *Gauss-Seidel scheme*: All components  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  are updated *sequentially*, one after the other, via the mapping  $\mathbf{T}$ ;
- iii) *Totally asynchronous scheme*: All components  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  are updated in a *totally asynchronous way* (in the sense of [22]), via the mapping  $\mathbf{T}$ . According to this scheme, some components  $\mathbf{x}_q$  may be updated more frequently than others and, when they are updated, a possibly *outdated* information on the other components can be used. Some variations of such a totally asynchronous scheme, e.g., including constraints on the maximum tolerable delay in the updating and on the use of the outdated information (which leads to the so-called *partially asynchronous algorithms*), can also be considered [22].

Observe that the latter algorithm contains, as special cases, the first two ones. In general, the above algorithms converge to the fixed-point of  $\mathbf{T}$  under different conditions [22], [24]. However, we can obtain a unified set of convergence conditions (not necessarily the mildest ones) by studying the contraction properties of mapping  $\mathbf{T}$  under a proper choice of the norm. To prove the convergence of the totally asynchronous algorithms, a useful norm is the so-called *block-maximum norm*, defined as follows. According to the partition  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  of  $\mathbf{x}$  and  $\mathbf{T} = (\mathbf{T}_q)_{q=1}^Q$ , with  $\mathbf{T}_q : \mathcal{X} \mapsto \mathcal{X}_q$ , let  $\|\cdot\|_q$  denote any vector norm on  $\mathbb{R}^{n_q}$  for each  $q$ , the block-maximum norm on  $\mathbb{R}^n$  is defined as [22], [24]

$$\|\mathbf{T}\|_{\text{block}} = \max_q \|\mathbf{T}_q\|_q. \quad (13)$$

The mapping  $\mathbf{T}$  is said to be a block-contraction with modulus  $\alpha \in [0, 1)$  if it is a contraction in the block-maximum norm with modulus  $\alpha$ . A unified set of convergence conditions for distributed algorithms based on mapping  $\mathbf{T}$  is given in the following theorem, whose proof follows the same steps as in

<sup>5</sup>For the sake of simplicity, we focus on mappings  $\mathbf{T}$  whose domain can be written as the Cartesian product of lower dimensional sets, associated to the partition of the mapping. For our purposes, this choice is enough, since the joint admissible strategy set of game  $\mathcal{G}$  satisfies this condition.

[20, Appendix A] and is omitted here because of the space limitation (see also [21]).

*Theorem 2: Given the mapping  $\mathbf{T} = (\mathbf{T}_q)_{q=1}^Q : \mathcal{X} \mapsto \mathcal{X}$ , with  $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_Q$ , assume that  $\mathbf{T}$  is a block-contraction with modulus  $\alpha \in [0, 1)$ . Then, the totally asynchronous algorithm (cf. [22]) based on the mapping  $\mathbf{T}$  asymptotically converges to the unique fixed-point of  $\mathbf{T}$ , for any set of initial conditions in  $\mathcal{X}$  and updating schedule.  $\square$*

Theorem 2 provides a unified set of convergence conditions for all the algorithms that are special cases of the totally asynchronous algorithm. Weaker conditions can still be obtained if one is not interested in a totally asynchronous implementation. For example, if only the Jacobi updating scheme is considered, to prove the contraction property of  $\mathbf{T}$ , one can use *any* arbitrary norm on  $\mathbb{R}^n$  [22], [24].

### C. Contraction Theory, Lyapunov Function, and Variational Inequality Problems

Contraction theory is not the only instrument available to prove the convergence of distributed algorithms to the fixed-point of a mapping  $\mathbf{T}$ . So far, we have seen that any mapping  $\mathbf{T}$  defines a dynamical system (see (11)). Hence, the convergence of distributed algorithms to the fixed-point of  $\mathbf{T}$  can be reformulated as the study of the globally asymptotic stability of the equilibrium of a proper dynamical system, based on  $\mathbf{T}$ . From this perspective, Lyapunov theory is a valuable instrument to study the system behavior and, as a by-product, the convergence of distributed algorithms [26], [35]. Indeed, the contraction property of mapping  $\mathbf{T}$  in the vector norm  $\|\cdot\|$  implies the existence of a valid Lyapunov function for the dynamical system in (11) [26], [35], given by  $V(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^*\|$ . This guarantees the convergence of a Jacobi scheme based on mapping  $\mathbf{T}$ . Interestingly, in the case of contraction of mapping  $\mathbf{T}$  in the block maximum norm (13), the Lyapunov function  $V(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^*\|_{\text{block}}$  can be interpreted as the *common* Lyapunov function of a set of interconnected dynamical systems, each of them associated to the partition  $\mathbf{x}_1, \dots, \mathbf{x}_Q$  of  $\mathbf{x}$  [36].

Finally, it is interesting to observe that the convergence to the fixed-point of a mapping  $\mathbf{T}$  can also be studied introducing a proper transformation of  $\mathbf{T}$  that preserves the set of the fixed-points. A useful tool to explore this direction is given by the variational inequality theory [39], [40] (see, e.g., [12], [19] and Section IV-A for an application of this framework to the multiuser waterfilling mapping).

We are now ready to apply the previous general framework to the multiuser waterfilling mapping in (9), as detailed next.

## IV. CONTRACTION PROPERTIES OF THE MULTIUSER WATERFILLING MAPPING

So far we have seen that a unified set of conditions guaranteeing the uniqueness of the NE and the convergence of totally asynchronous algorithms to the fixed-point Nash equilibria of game  $\mathcal{G}$  can be obtained deriving conditions for the multiuser waterfilling mapping in (9) to be a contraction in a proper block-maximum norm (see Theorems 1 and 2). In this section we then provide a contraction theorem for

the multiuser waterfilling operator. Our result is based on the interpretation of MIMO waterfilling operator as a matrix projection onto the convex set of feasible strategies of the users. This result is also useful to obtain a unified view of, apparently different, techniques used in the literature to study the uniqueness of the NE and the convergence of alternative waterfilling based algorithms in the rate-maximization game over *SISO frequency-selective* Gaussian interference channels [7]-[20]. To show this interesting relationship, we start from an overview of the main properties of the Nash equilibria of game  $\mathcal{G}$  in the case of SISO frequency-selective Gaussian interference channels, as obtained in [14], [19], [20], [12], [16], and [7], [15], and then we consider the more general MIMO case.

### A. Frequency-Selective Gaussian Interference Channels

In the case of block transmission over SISO frequency-selective channels [7]-[20], each channel matrix  $\mathbf{H}_{r,q} \in \mathbb{C}^{N \times N}$  becomes a Toeplitz circulant matrix and  $n_{T_q} = n_{R_q} = N$ , where  $N$  is the length of the transmitted block (see, e.g., [18]). This leads to the eigendecomposition  $\mathbf{H}_{r,q} = \mathbf{W}\mathbf{D}_{r,q}\mathbf{W}^H$ , where  $\mathbf{W} \in \mathbb{C}^{N \times N}$  is the normalized IFFT matrix, i.e.,  $[\mathbf{W}]_{i,j} \triangleq e^{j2\pi(i-1)(j-1)/N}/\sqrt{N}$  for  $i, j = 1, \dots, N$  and  $\mathbf{D}_{r,q}$  is a  $N \times N$  diagonal matrix, where  $[\mathbf{D}_{r,q}]_{kk} \triangleq H_{r,q}(k)$  is the frequency-response of the channel between source  $r$  and destination  $q$  at carrier  $k$ , and  $\mathbf{R}_{n_q} = \text{Diag}(\sigma_q^2(1), \dots, \sigma_q^2(N))$ . Under this setup, the matrix game  $\mathcal{G}$  in (4) reduces to a simpler power control game, where the strategy of each user  $q$  becomes the power allocation  $\mathbf{p}_q = [p_q(1), \dots, p_q(N)]^T$  over the  $N$  carriers and the admissible strategy set in (5) reduces to<sup>6</sup>

$$\mathcal{P}_q \triangleq \left\{ \mathbf{x} \in \mathbb{R}_+^N : \sum_{k=1}^N x_k = P_q, \quad \forall k \in \{1, \dots, N\} \right\}, \quad (14)$$

and  $\mathcal{P} \triangleq \mathcal{P}_1 \times \dots \times \mathcal{P}_Q$ . It follows that the optimal strategy at any NE must satisfy the simultaneous multiuser waterfilling equation:

$$\mathbf{p}_q^* = \mathbf{wf}_q(\mathbf{p}_{-q}^*), \quad \forall q \in \Omega, \quad (15)$$

where  $\mathbf{p}_{-q} \triangleq (\mathbf{p}_r)_{r \in \Omega, r \neq q}$ , and the waterfilling operator  $\mathbf{wf}_q(\cdot)$  becomes [19]

$$[\mathbf{wf}_q(\mathbf{p}_{-q})]_k \triangleq \left( \mu_q - \frac{\sigma_q^2(k) + \sum_{r \neq q} |H_{r,q}(k)|^2 p_r(k)}{|H_{q,q}(k)|^2} \right)^+, \quad (16)$$

for  $k \in \{1, \dots, N\}$ , with the waterlevel  $\mu_q$  chosen to satisfy the power constraint  $\sum_{k=1}^N p_q(k) = P_q$ .

Different approaches have been proposed in the literature to study the properties of the Nash equilibria in (15), each time obtaining milder conditions for the uniqueness and the convergence of distributed algorithms [7]-[20]. We provide in the following a unified view of the techniques used in the cited papers, based on the mathematical framework described in Section III.

<sup>6</sup>In the transmissions over frequency-selective channels, in addition to the power constraints as in (14), it may be useful to introduce spectral mask constraints, in order to impose radiation limits over some (licensed) bands [12], [19], [20], [14]. Interestingly, all the results presented on this section are valid also in the presence of spectral mask constraints, as proved in [12], [19], [20], [14].

**Approach #1: Multiuser waterfilling as a projector [14], [19]** - We introduce first the following intermediate definitions. For any given  $\mathbf{p}_{-q}$ , let  $\mathbf{insr}_q(\mathbf{p}_{-q})$  be the  $N$ -dimensional vector defined as

$$[\mathbf{insr}_q(\mathbf{p}_{-q})]_k \triangleq \frac{\sigma_q^2(k) + \sum_{r \neq q} |H_{r,q}(k)|^2 p_r(k)}{|H_{q,q}(k)|^2}, \quad (17)$$

with  $k \in \{1, \dots, N\}$ . In order to apply Theorem 2, we introduce a proper block-maximum norm for the multiuser waterfilling mapping  $\mathbf{wf} = (\mathbf{wf}_q(\mathbf{p}_{-q}))_{q \in \Omega}$  in (16) (cf. Section III-B). Given some  $\mathbf{w} \triangleq [w_q, \dots, w_Q]^T > \mathbf{0}$ , let  $\|\cdot\|_{2,\text{block}}^{\mathbf{w}}$  denote the (vector) block-maximum norm, defined as [22]

$$\|\mathbf{wf}(\mathbf{p})\|_{2,\text{block}}^{\mathbf{w}} \triangleq \max_{q \in \Omega} \frac{\|\mathbf{wf}_q(\mathbf{p}_{-q})\|_2}{w_q}, \quad (18)$$

where  $\|\cdot\|_2$  is the Euclidean norm. Let  $\|\cdot\|_{\infty,\text{vec}}^{\mathbf{w}}$  be the vector weighted maximum norm, defined as [41]

$$\|\mathbf{x}\|_{\infty,\text{vec}}^{\mathbf{w}} \triangleq \max_{q \in \Omega} \frac{|x_q|}{w_q}, \quad \mathbf{w} > \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^Q, \quad (19)$$

and let  $\|\cdot\|_{\infty,\text{mat}}^{\mathbf{w}}$  denote the matrix norm induced by  $\|\cdot\|_{\infty,\text{vec}}^{\mathbf{w}}$ , given by [41]

$$\|\mathbf{A}\|_{\infty,\text{mat}}^{\mathbf{w}} \triangleq \max_q \frac{1}{w_q} \sum_{r=1}^Q |[\mathbf{A}]_{qr}| w_r, \quad \mathbf{A} \in \mathbb{R}^{Q \times Q}. \quad (20)$$

We also introduce the nonnegative matrix  $\mathbf{S}^{\max} \in \mathbb{R}_+^{Q \times Q}$ , defined as

$$[\mathbf{S}^{\max}]_{qr} \triangleq \begin{cases} \max_{k \in \{1, \dots, N\}} \frac{|H_{qr}(k)|^2}{|H_{qq}(k)|^2}, & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \quad (21)$$

Using the above definitions and denoting by  $[\mathbf{x}_0]_{\mathcal{X}} = \text{argmin}_{\mathbf{z} \in \mathcal{X}} \|\mathbf{z} - \mathbf{x}_0\|_2$  the Euclidean projection of vector  $\mathbf{x}_0$  onto the convex set  $\mathcal{X}$ , in [14], [19] we proved the following.

*Lemma 1 (Waterfilling as a projector):* The waterfilling operator  $\mathbf{wf}_q(\mathbf{p}_{-q})$  in (16) can be equivalently written as

$$\mathbf{wf}_q(\mathbf{p}_{-q}) = [-\mathbf{insr}_q(\mathbf{p}_{-q})]_{\mathcal{P}_q}, \quad (22)$$

where  $\mathcal{P}_q$  and  $\mathbf{insr}_q(\cdot)$  are defined in (14) and (17), respectively.  $\square$

It follows from Lemma 1 that the Nash equilibria in (15) can be alternatively obtained as the fixed-points of the mapping defined in (22)

$$\mathbf{p}_q^* = [-\mathbf{insr}_q(\mathbf{p}_{-q}^*)]_{\mathcal{P}_q}, \quad \forall q \in \Omega. \quad (23)$$

Lemma 1 is also the key result to study contraction properties of mapping  $\mathbf{wf}$  and thus, based on (23), to derive conditions for the uniqueness of the NE and convergence of distributed algorithms. The main result is summarized in the following theorem that comes from [19, Proposition 2].<sup>7</sup>

<sup>7</sup>We refer the interested reader to [19], [20] for more general results on contraction properties of the  $\mathbf{wf}$  mapping.

**Theorem 3** (Contraction property of the mapping  $\mathbf{wf}$ ):

Given  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$ , the mapping  $\mathbf{wf}$  defined in (16) is Lipschitz continuous on  $\mathcal{P}$ :

$$\left\| \mathbf{wf}(\mathbf{p}^{(1)}) - \mathbf{wf}(\mathbf{p}^{(2)}) \right\|_{2,\text{block}}^{\mathbf{w}} \leq \left\| \mathbf{S}^{\max} \right\|_{\infty,\text{mat}}^{\mathbf{w}} \times \left\| \mathbf{p}^{(1)} - \mathbf{p}^{(2)} \right\|_{2,\text{block}}^{\mathbf{w}}, \quad (24)$$

$\forall \mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P}$ , where  $\|\cdot\|_{2,\text{block}}^{\mathbf{w}}$ ,  $\|\cdot\|_{\infty,\text{mat}}^{\mathbf{w}}$ , and  $\mathbf{S}^{\max}$  are defined in (18), (20), and (21), respectively. Furthermore, if

$$\left\| \mathbf{S}^{\max} \right\|_{\infty,\text{mat}}^{\mathbf{w}} < 1, \quad (25)$$

for some  $\mathbf{w} > \mathbf{0}$ , then mapping  $\mathbf{wf}$  is a block-contraction with modulus  $\left\| \mathbf{S}^{\max} \right\|_{\infty,\text{mat}}^{\mathbf{w}}$ .  $\square$

Given Theorem 3, it follows from Theorem 1 and Theorem 2, that condition  $\left\| \mathbf{S}^{\max} \right\|_{\infty,\text{mat}}^{\mathbf{w}} < 1$  is sufficient to guarantee the uniqueness of the NE of the game as well as the convergence of totally asynchronous algorithm based on the waterfilling mapping  $\mathbf{wf}$  in (16) [19], [20].

**Approach #2: Multiuser waterfilling as solution of an Affine VI [12]** - In [12], the authors established an interesting reformulation of the rate maximization game as a linear complementarity problem (LCP) [39]. More specifically, they proved that the nonlinear system of KKT optimality conditions of the  $q$ -th user convex problem in  $\mathcal{G}$ , given by [where  $a \perp b$  means that the two scalars  $a$  and  $b$  are orthogonal, i.e.,  $a \cdot b = 0$ ]<sup>8</sup>

$$\begin{aligned} -\frac{|H_{qq}(k)|^2}{\sigma_q^2(k) + \sum_{r=1}^Q |H_{rq}(k)|^2 p_r(k)} + \mu_q - \nu_{q,k} &= 0, & \forall k, \\ 0 \leq \mu_q \perp \left( P_q - \sum_{k=1}^N p_q(k) \right) &\geq 0, \\ 0 \leq \nu_{q,k} \perp p_q(k) &\geq 0, & \forall k, \end{aligned} \quad (26)$$

is equivalent to [12, Proposition 1]:

$$\begin{aligned} 0 \leq p_q(k) \perp \left( \frac{\sigma_q^2(k)}{|H_{qq}(k)|^2} + \sum_{r=1}^Q \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2} p_r(k) + \lambda_q \right) &\geq 0, \\ \lambda_q = \text{free}, \quad \sum_{k=1}^N p_q(k) &= P_q. \end{aligned} \quad (27)$$

As observed in [12], (27) for all  $q \in \Omega$  represents the KKT conditions of the Affine VI (AVI)  $(\mathcal{P}, \tilde{\sigma}, \mathbf{M})$  defined by the polyhedral set  $\mathcal{P}$  and the affine mapping  $\mathbf{p} \mapsto \tilde{\sigma} + \mathbf{M}\mathbf{p}$  (see [39] for more details on the AVI problems), where  $\mathbf{p} \triangleq [p_1^T, \dots, p_Q^T]^T$ ,  $\tilde{\sigma} \triangleq [\tilde{\sigma}_1^T, \dots, \tilde{\sigma}_Q^T]^T$ , and  $\mathbf{M}$  is a block partition matrix  $\mathbf{M} \triangleq [\mathbf{M}_{rq}]_{r,q \in \Omega}$ , with  $\mathbf{p}_q = [p_q(1), \dots, p_q(N)]^T$ ,  $\tilde{\sigma}_q = [\sigma_q^2(1)/|H_{qq}(1)|^2, \dots, \sigma_q^2(N)/|H_{qq}(N)|^2]^T$  and  $\mathbf{M}_{qr} \triangleq \text{Diag}(|H_{rq}(1)|^2/|H_{qq}(1)|^2, \dots, |H_{rq}(N)|^2/|H_{qq}(N)|^2)$ , for  $r, q \in \Omega$ .

It follows that the vector  $\mathbf{p}^* \in \mathcal{P}$  is a NE of the game  $\mathcal{G}$  if and only if it satisfies the AVI  $(\mathcal{P}, \tilde{\sigma}, \mathbf{M})$  [39], [40]:

$$(\mathbf{p} - \mathbf{p}^*)^T (\tilde{\sigma} + \mathbf{M}\mathbf{p}^*) \geq 0, \quad \forall \mathbf{p} \in \mathcal{P}. \quad (28)$$

<sup>8</sup>Observe that in [12], the authors considered equivalently for each user  $q \in \Omega$  the power constraint  $\sum_{k=1}^N p_q(k) \leq P_q$ , rather than  $\sum_{k=1}^N p_q(k) = P_q$ , as we did in (14).

Building on this result and the properties of AVI problems (cf. [39], [40]), the authors in [12] derived sufficient conditions for the uniqueness of the Nash equilibrium and the global convergence of synchronous sequential IWFA.

It can be shown that the AVI  $(\mathcal{P}, \tilde{\sigma}, \mathbf{M})$  in (28) is equivalent to the fixed-point equation in (23) [40], [39], establishing the link between the solutions to (28)—the fixed-points of the waterfilling mapping  $\mathbf{wf}$  in (16)—and the interpretation of the mapping  $\mathbf{wf}$  as a projection (Lemma 1), as given in [14], [19].<sup>9</sup> In fact, the convergence conditions obtained in [12] for the synchronous sequential IWFA coincide with (25) (for a proper choice of vector  $\mathbf{w}$  [19]).<sup>10</sup> Observe that they are a special case of those obtained in [20, Corollary 2].

**Approach #3: Multiuser waterfilling as a piecewise affine function [16], [40, Ch. 4]** - In [16], the authors proved global convergence of different waterfilling based algorithms building on the key result that the waterfilling mapping  $\mathbf{wf} : \mathcal{P} \mapsto \mathcal{P}$  can be equivalently written as a piecewise affine (PA) function on  $\mathbb{R}^{QN}$  [40, Ch. 4]. In fact, the result in [16] follows the same steps as Propositions 4.1.1 and 4.2.2 in [40, Ch. 4] and, interestingly, can be obtained directly from our interpretation of the  $\mathbf{wf}$  as a projection (Lemma 1) and some properties of the PA functions in [40, Ch. 4], as detailed next. We introduce the following intermediate definitions first.

*Definition 2* ([40, Def. 4.1.3]): A continuous function  $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$  is said to be piecewise affine (PA) if there exists a finite family of affine functions  $\{\mathbf{f}_k(\mathbf{x}) = \mathbf{A}_k \mathbf{x} + \mathbf{b}_k\}_{k=1}^K$  for some positive integer  $K$  and  $\{(\mathbf{A}_k, \mathbf{b}_k)\}_{k=1}^K$ , with each  $\mathbf{f}_k : \mathbb{R}^n \mapsto \mathbb{R}^m$ , such that for all  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{f}(\mathbf{x}) \in \{\mathbf{f}_1(\mathbf{x}), \dots, \mathbf{f}_K(\mathbf{x})\}$ .  $\square$

PA functions have many interesting properties (we refer the interested reader to [40, Ch. 4] for an in-depth study of the theory of PA functions). Here, we are interested in the following one, which follows directly from [40, Proposition 4.2.2 (c)].

*Lemma 2:* Any PA map  $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$  is globally Lipschitz continuous on  $\mathbb{R}^n$ :

$$\left\| \mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(2)}) \right\|_{\text{vec}} \leq \alpha \left\| \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \right\|_{\text{vec}}, \quad \forall \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{R}^n, \quad (29)$$

with Lipschitz constant  $\alpha \triangleq \max_{k \in \{1, \dots, K\}} \left\| \mathbf{A}_k \right\|_{\text{mat}}$ , where  $\|\cdot\|_{\text{vec}}$  is any vector norm and  $\|\cdot\|_{\text{mat}}$  is the matrix norm induced by  $\|\cdot\|_{\text{vec}}$ .  $\square$

The link between our interpretation of the  $\mathbf{wf}$  mapping as a projector (Lemma 1) and the interpretation of  $\mathbf{wf}$  as PA map ([16, Theorem 5]) is given by the following [40, Prop. 4.1.4].

*Lemma 3:* Let  $\mathcal{X}$  be a polyhedral set in  $\mathbb{R}^n$ .<sup>11</sup> Then, the Euclidean projector onto  $\mathcal{X}$  is a PA function on  $\mathbb{R}^n$ .  $\square$

According to Lemma 1, for any given  $\mathbf{p} \geq \mathbf{0}$ , the waterfilling  $\mathbf{wf}(\mathbf{p})$  in (16) is the Euclidean projector of vector  $-\mathbf{insr}(\mathbf{p}) \triangleq -[\mathbf{insr}_1(\mathbf{p}_{-1})^T, \dots, \mathbf{insr}_Q(\mathbf{p}_{-Q})^T]^T$  onto the

<sup>9</sup>Papers [18] and [19] have been originally submitted on September 2004 (see also [14]).

<sup>10</sup>Recall that condition (25) is valid also for the convergence of the asynchronous IWFA (Theorem 2) [20].

<sup>11</sup>Roughly speaking, a polyhedral is the intersection of a finite number of halfspaces and hyperplanes (see, e.g., [42, Ch. 2.2.4]).

convex set  $\mathcal{P}$ , which is a *polyhedral set*. It then follows from Lemma 3 that  $\mathbf{wf}(\mathbf{p})$  is a PA function, i.e., there exists a finite family of affine functions  $\{\mathbf{f}_k(\mathbf{p}) = \mathbf{A}_k\mathbf{p} + \mathbf{b}_k\}_{k=1}^K$  such that, for every  $\mathbf{p} \geq \mathbf{0}$  we have

$$\mathbf{wf}(\mathbf{p}) = \mathbf{A}_k\mathbf{p} + \mathbf{b}_k, \quad \text{for some } k \in \{1, \dots, K\}, \quad (30)$$

which coincides with the result in [16, Theorem 5]. The expression of the affine pieces  $\{\mathbf{A}_k, \mathbf{b}_k\}_{k=1}^K$  of  $\mathbf{wf}(\mathbf{p})$  can be obtained exploring the structure of the waterfilling solution (16). We omit the details here because of space limitations (see ([16, Theorem 5])).

Contraction properties of the  $\mathbf{wf}$  operator interpreted as PA map on  $\mathbb{R}^{QN}$  as stated in [16, Theorem 7] are a direct consequence of properties of the PA functions. In particular, it follows from Lemma 2 that, if

$$\alpha = \max_{k \in \{1, \dots, K\}} \|\mathbf{A}_k\|_{\text{mat}} < 1, \quad (31)$$

then the  $\mathbf{wf}$  mapping is a contraction in the vector norm  $\|\cdot\|_{\text{vec}}$ . Exploring different vector norms, one can easily obtain different sufficient conditions for the uniqueness of the NE of the game and the convergence of distributed algorithms, based on the  $\mathbf{wf}$  mapping. Observe that, according to Theorem 2, if a block-maximum norm is used in (29) (see also Theorem 3 and Theorem 5 in Section IV-B2), then condition (31) guarantees also the global convergence of *totally asynchronous* algorithms [20], [16].

**Approach #4: Multiuser waterfilling via the max-lemma [7], [15]** - In [15], the authors, among all, derived conditions for the global convergence of the synchronous sequential and simultaneous iterative waterfilling algorithms (implying also the uniqueness of the NE). Conditions in [15] generalize those obtained in [7] for the two-player game to the case of arbitrary number of players. We show now that results in [15] come from Theorem 2 as special case, if a proper vector norm is chosen. To this end, we introduce first the following intermediate definitions and results.

A key point in the proof of convergence in [15] is given by the following max-lemma [15, Lemma 1].

*Lemma 4:* Let  $f : \mathbb{R} \mapsto \mathbb{R}$  and  $g : \mathbb{R} \mapsto \mathbb{R}$  be a non-decreasing and non-increasing functions on  $\mathbb{R}$ , respectively. If there exists a unique  $x^*$  such that  $f(x^*) = g(x^*)$ , and function  $f$  and  $g$  are strictly increasing and strictly decreasing at  $x = x^*$ , then

$$x^* = \underset{x}{\operatorname{argmin}} \{ \max \{ f(x), g(x) \} \}. \quad (32)$$

□

Let us introduce the following error dynamic, as defined in [15]:

$$e_q(n+1) \triangleq \max \left\{ \sum_k \left[ p_q^{(n+1)}(k) - p_q^{(n)}(k) \right]^+, \sum_k \left[ p_q^{(n+1)}(k) - p_q^{(n)}(k) \right]^- \right\}, \quad (33)$$

with  $n \in \mathbb{N}_+ = \{0, 1, 2, \dots\}$ , where  $\mathbf{p}_q^{(n)} = [p_q^{(n)}(1), \dots, p_q^{(n)}(N)]^T$  is the vector of the power allocation

of user  $q \in \Omega$ , generated at the discrete time  $n$  by the sequential or simultaneous IWFA [starting from any arbitrary feasible point  $\mathbf{p}^{(0)}$ ],  $[x]^+ \triangleq \max(0, x)$ , and  $[x]^- = \max(0, -x)$ . Using Lemma 4 and following the same steps as in [15], results in [15] can be restated in terms of the error vector  $\mathbf{e}^{(n+1)} \triangleq [e_1^{(n+1)}, \dots, e_Q^{(n+1)}]$  as:

$$\|\mathbf{e}^{(n+1)}\|_{\infty, \text{vec}} \leq \alpha \|\mathbf{e}^{(n)}\|_{\infty, \text{vec}}, \quad \forall n \in \mathbb{N}_+, \quad (34)$$

where  $\alpha \triangleq (Q-1) \max_{k, r \neq q} \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}$ , and  $\|\cdot\|_{\infty, \text{vec}}$  denotes the  $l_\infty$  norm [see (19) with  $\mathbf{w} = \mathbf{1}$ ]. It follows from (34) that, under  $\alpha < 1$ , both synchronous sequential and simultaneous IWFAs globally converge to the unique NE of the game [7], [15]. Observe that this condition implies the contraction of the waterfilling mapping as given in Theorem 3, showing the more generality of our sufficient condition (25) than that in [7], [15].

**Extension:** We generalize now the results in [7], [15], so that we can use Theorem 2 and enlarge the convergence conditions of [7], [15], making them to coincide with (25) and valid also for the asynchronous IWFA [20]. To this end, we introduce a new vector norm, as detailed next.

Inspired by (33), we introduce the following norm:

$$\|\mathbf{x}\|_{1, \infty, \text{vec}} \triangleq \max \left\{ \|(\mathbf{x})^+\|_{1, \text{vec}}, \|(\mathbf{x})^-\|_{1, \text{vec}} \right\}, \quad \mathbf{x} \in \mathbb{R}^N, \quad (35)$$

where  $\|\cdot\|_{1, \text{vec}}$  denotes the  $l_1$  norm [41]. Some properties of  $\|\cdot\|_{1, \infty, \text{vec}}$  are listed in the following lemma (we omit the proof because of space limitation).

*Lemma 5:* The norm  $\|\cdot\|_{1, \infty, \text{vec}}$  in (35) is a valid vector norm (in the sense that it satisfies the axioms of a norm [41]). Moreover, the following nonexpansion property holds true:<sup>12</sup>

$$\left\| (\mu_x \mathbf{1} - \mathbf{x}_0)^+ - (\mu_y \mathbf{1} - \mathbf{y}_0)^+ \right\|_{1, \infty, \text{vec}} \leq \|\mathbf{x}_0 - \mathbf{y}_0\|_{1, \infty, \text{vec}}, \quad \forall \mathbf{x}_0, \mathbf{y}_0 \in \mathbb{R}_+^N, \quad (36)$$

where the waterlevels  $\mu_x$  and  $\mu_y$  satisfy  $\mathbf{1}^T (\mu_x \mathbf{1} - \mathbf{x}_0)^+ = \mathbf{1}^T (\mu_y \mathbf{1} - \mathbf{y}_0)^+ = P_T$ , (with  $P_T$  an arbitrary nonnegative number), and  $\mathbf{1}$  denotes the  $N$ -dimensional vector of all ones. □

Interestingly, Lemma 5 provides the nonexpansion property of the (single-user) waterfilling solution in the vector norm  $\|\cdot\|_{1, \infty, \text{vec}}$ . It also represents the key result to prove the contraction property of the multiuser waterfilling mapping  $\mathbf{wf}$  in (16) in the block-maximum norm  $\|\cdot\|_{1, \infty, \text{block}}^{\mathbf{w}}$  associated to  $\|\cdot\|_{1, \infty, \text{vec}}$ , defined, for each  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$ , as

$$\|\mathbf{wf}(\mathbf{p})\|_{1, \infty, \text{block}}^{\mathbf{w}} \triangleq \max_{q \in \Omega} \frac{\|\mathbf{wf}_q(\mathbf{p}-q)\|_{1, \infty, \text{vec}}}{w_q}. \quad (37)$$

The main result is stated next (the proof is based on Lemma 5 and follows similar steps of that in [20, Appendix A]; see also Theorem 5 in Section IV-B2).

<sup>12</sup>Interestingly, one can prove that the nonexpansion property as stated in (36) also holds true if in (36) the norm  $\|\cdot\|_{1, \infty, \text{vec}}$  is replaced by the  $l_1$  norm.

**Theorem 4** (Contraction property of mapping  $\mathbf{wf}$ ): *Given  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$ , the mapping  $\mathbf{wf}$  defined in (16) is Lipschitz continuous on  $\mathcal{P}$ :*

$$\left\| \mathbf{wf}(\mathbf{p}^{(1)}) - \mathbf{wf}(\mathbf{p}^{(2)}) \right\|_{1, \infty, \text{block}}^{\mathbf{w}} \leq \left\| \mathbf{S}^{\max} \right\|_{\infty, \text{mat}}^{\mathbf{w}} \times \left\| \mathbf{p}^{(1)} - \mathbf{p}^{(2)} \right\|_{1, \infty, \text{block}}^{\mathbf{w}}, \quad (38)$$

$\forall \mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P}$ , where  $\|\cdot\|_{1, \infty, \text{block}}^{\mathbf{w}}$ ,  $\|\cdot\|_{\infty, \text{mat}}^{\mathbf{w}}$  and  $\mathbf{S}^{\max}$  are defined in (37), (20), and (21), respectively.  $\square$

Comparing Theorem 3 with Theorem 4, one infers that both theorems provide the same sufficient conditions for the waterfilling mapping  $\mathbf{wf}$  to be a contraction and thus the same sufficient conditions guaranteeing the uniqueness of the NE and the convergence of asynchronous IWFAs [19], [20].

### B. MIMO Gaussian Interference Channels

In this section, we generalize our interpretation of the waterfilling projector in the frequency-selective case to the MIMO multiuser case. For the sake of simplicity, we concentrate on MIMO systems whose direct channel matrices  $\mathbf{H}_{qq}$  are square and nonsingular. The more general case is much more involved and goes beyond the scope of the present paper; it has been considered in [21].

1) *Multiuser waterfilling in Gaussian MIMO interference channels*: We first introduce the following intermediate result.

**Proposition 1**: *Given  $\mathbf{R}_n \succ \mathbf{0}$ ,  $\mathbf{H} \in \mathbb{C}^{n \times n}$ , and  $P_T > 0$ , define the following two convex optimization problems:*

$$(P1): \quad \begin{array}{ll} \underset{\mathbf{X} \succeq \mathbf{0}}{\text{maximize}} & \log \det \left( \mathbf{R}_n + \mathbf{H}\mathbf{X}\mathbf{H}^H \right) \\ \text{subject to} & \text{Tr}\{\mathbf{X}\} \leq P_T, \end{array} \quad (39)$$

and

$$(P2): \quad \begin{array}{ll} \underset{\mathbf{X} \succeq \mathbf{0}}{\text{minimize}} & \|\mathbf{X} - \mathbf{X}_0\|_F^2 \\ \text{subject to} & \text{Tr}\{\mathbf{X}\} = P_T. \end{array} \quad (40)$$

If  $\mathbf{X}_0$  in (P2) is chosen as  $\mathbf{X}_0 = -(\mathbf{H}^H \mathbf{R}_n^{-1} \mathbf{H})^{-1}$ , then both problems (P1) and (P2) have the same unique solution.

*Proof*: Problem (P1) [and (P2)] is convex and admits a unique solution, since the objective function is strictly concave (and strictly convex) on  $\mathbf{X} \succeq \mathbf{0}$ . The Lagrangian function  $\mathcal{L}$  associated to (39) is

$$\mathcal{L} = -\log \det \left( \mathbf{R}_n + \mathbf{H}\mathbf{X}\mathbf{H}^H \right) - \text{Tr}(\Psi \mathbf{X}) + \lambda (\text{Tr}(\mathbf{X}) - P_T), \quad (41)$$

which leads to the following KKT optimality conditions (Slater's conditions are satisfied [42, Ch. 5.9.1]):

$$-\mathbf{H}^H \left( \mathbf{R}_n + \mathbf{H}\mathbf{X}\mathbf{H}^H \right)^{-1} \mathbf{H} + \lambda \mathbf{I} = \Psi, \quad (42)$$

$$\Psi \succeq \mathbf{0}, \quad \mathbf{X} \succeq \mathbf{0}, \quad \text{Tr}(\Psi \mathbf{X}) = 0, \quad (43)$$

$$\lambda \geq 0, \quad \lambda (\text{Tr}(\mathbf{X}) - P_T) = 0, \quad \text{Tr}(\mathbf{X}) \leq P_T. \quad (44)$$

First of all, observe that  $\lambda$  must be positive. Otherwise, (42) would lead to

$$\mathbf{0} \succ -\mathbf{H}^H \left( \mathbf{R}_n + \mathbf{H}\mathbf{X}\mathbf{H}^H \right)^{-1} \mathbf{H} = \Psi \succeq \mathbf{0}, \quad (45)$$

which cannot be true. We rewrite now (42)-(44) in a more convenient form. To this end, we introduce

$$\mathbf{X}_0 \triangleq -(\mathbf{H}^H \mathbf{R}_n^{-1} \mathbf{H})^{-1} \prec \mathbf{0}, \quad (46)$$

so that

$$\mathbf{H}^H \left( \mathbf{R}_n + \mathbf{H}\mathbf{X}\mathbf{H}^H \right)^{-1} \mathbf{H} = (\mathbf{X} - \mathbf{X}_0)^{-1} \succ \mathbf{0}. \quad (47)$$

Then, using the fact that  $\lambda > 0$  and absorbing in (42)-(43) the slack variable  $\Psi$ , system (42)-(44) can be rewritten as

$$\mathbf{X} \left[ -(\mathbf{X} - \mathbf{X}_0)^{-1} + \lambda \mathbf{I} \right] = \mathbf{0}, \quad (48)$$

$$\mathbf{X} \succeq \mathbf{0}, \quad -(\mathbf{X} - \mathbf{X}_0)^{-1} + \lambda \mathbf{I} \succeq \mathbf{0}, \quad (49)$$

$$\lambda > 0, \quad \text{Tr}(\mathbf{X}) = P_T, \quad (50)$$

where in (48) we have used the following fact [38, fact 8.10.3]

$$\text{Tr}(\Psi \mathbf{X}) = 0 \quad \Leftrightarrow \quad \Psi \mathbf{X} = \mathbf{0}, \quad \forall \Psi, \mathbf{X} \succeq \mathbf{0}. \quad (51)$$

Since  $\lambda > 0$ , (48)-(50) become

$$\mathbf{X} \left[ (\mathbf{X} - \mathbf{X}_0) - \frac{1}{\lambda} \mathbf{I} \right] = \mathbf{0}, \quad (52)$$

$$\mathbf{X} \succeq \mathbf{0}, \quad -(\mathbf{X} - \mathbf{X}_0)^{-1} + \lambda \mathbf{I} \succeq \mathbf{0}, \quad (53)$$

$$\lambda > 0, \quad \text{Tr}(\mathbf{X}) = P_T. \quad (54)$$

We show now that (52)-(54) is equivalent to

$$\mathbf{X} [(\mathbf{X} - \mathbf{X}_0) + \mu \mathbf{I}] = \mathbf{0}, \quad (55)$$

$$\mathbf{X} \succeq \mathbf{0}, \quad \mathbf{X} - \mathbf{X}_0 + \mu \mathbf{I} \succeq \mathbf{0}, \quad (56)$$

$$\mu = \text{free}, \quad \text{Tr}(\mathbf{X}) = P_T. \quad (57)$$

(52)-(54)  $\Rightarrow$  (55)-(57): Let  $(\mathbf{X}, \lambda)$  be a solution of (52)-(54). A solution of (55)-(57) is obtained using  $(\mathbf{X}, \mu)$ , with  $\mu = -\frac{1}{\lambda}$ . (55)-(57)  $\Rightarrow$  (52)-(54): Let  $(\mathbf{X}, \mu)$  be a solution of (55)-(57). It must be  $\mu < 0$ ; otherwise, since  $\mathbf{X} - \mathbf{X}_0 \succ \mathbf{0}$  [see (46)], (55) would lead to  $\mathbf{X} = \mathbf{0}$ , which contradicts the power constraint in (57). Setting  $\lambda = -\frac{1}{\mu}$ , it is easy to check that  $(\mathbf{X}, \lambda)$  satisfies (52)-(54).

The system (55)-(57) represents the KKT optimality conditions of problem (40) with  $\mathbf{X}_0$  defined in (46); which completes the proof.  $\blacksquare$

Denoting by  $[\mathbf{X}_0]_{\mathcal{Q}_q}$  the matrix projection of  $\mathbf{X}_0$  with respect to the Frobenius norm onto the set  $\mathcal{Q}_q$  defined in (5)—the solution to problem (P2) in (40) with  $P_T = P_q$ —and using Proposition 1 we have directly the following.<sup>13</sup>

**Lemma 6**: *The waterfilling operator  $\mathbf{WF}_q(\mathbf{Q}_{-q})$  in (9) can be equivalently written as*

$$\mathbf{WF}_q(\mathbf{Q}_{-q}) = \left[ -(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1} (\mathbf{Q}_{-q}) \mathbf{H}_{qq})^{-1} \right]_{\mathcal{Q}_q}, \quad (58)$$

where  $\mathcal{Q}_q$  is defined in (5).  $\square$

Comparing (10) with (58), it is straightforward to see that all the Nash equilibria of game  $\mathcal{G}$  can be alternatively obtained as the fixed-points of the mapping defined in (58):

$$\mathbf{Q}_q^* = \left[ -(\mathbf{H}_{qq}^H \mathbf{R}_{-q}^{-1} (\mathbf{Q}_{-q}^*) \mathbf{H}_{qq})^{-1} \right]_{\mathcal{Q}_q}, \quad \forall q \in \Omega. \quad (59)$$

<sup>13</sup>A more general expression of the waterfilling projection valid for the general case of singular (possibly) rectangular channel matrices is given in [21].

*Remark 5 - Nonexpansive property of the MIMO waterfilling operator.* Thanks to the interpretation of MIMO waterfilling in (9) as a projector, one can obtain the following nonexpansive property of the waterfilling operator that will be used in the next section to derive the contraction properties of the MIMO waterfilling mapping.

*Lemma 7:* Let  $[\cdot]_{\mathcal{Q}_q}$  denote the matrix projection onto the convex set  $\mathcal{Q}_q$  with respect to the Frobenius norm, as defined in (40). Then,  $[\cdot]_{\mathcal{Q}_q}$  satisfies the following nonexpansive property:

$$\left\| [\mathbf{X}]_{\mathcal{Q}_q} - [\mathbf{Y}]_{\mathcal{Q}_q} \right\|_F \leq \|\mathbf{X} - \mathbf{Y}\|_F, \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n_{T_q} \times n_{T_q}}. \quad (60)$$

□

2) *Contraction property of MIMO multiuser waterfilling:* Building on Lemmas 6 and 7, we derive now sufficient conditions for the waterfilling mapping to be a contraction, under a proper norm. Our result is the natural extension of Theorem 3 to the MIMO case.

As in the SISO case, we define first an appropriate block-maximum norm for the multiuser waterfilling mapping. Given

$$\mathbf{WF}(\mathbf{Q}) = (\mathbf{WF}_q(\mathbf{Q}_{-q}))_{q \in \Omega} : \mathcal{Q} \mapsto \mathcal{Q}, \quad (61)$$

where  $\mathcal{Q} = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_Q$ , with  $\mathcal{Q}_q$  and  $\mathbf{WF}_q(\mathbf{Q}_{-q})$  defined in (5) and (58), respectively, we introduce the following block-maximum norm on  $\mathbb{C}^{n \times n}$ , with  $n = n_{T_1} + \dots + n_{T_Q}$ , defined as [22]

$$\|\mathbf{WF}(\mathbf{Q})\|_{F, \text{block}}^{\mathbf{w}} \triangleq \max_{q \in \Omega} \frac{\|\mathbf{WF}_q(\mathbf{Q}_{-q})\|_F}{w_q}, \quad (62)$$

where  $\|\cdot\|_F$  is the Frobenius norm and  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$  is any positive weight vector. Finally, let  $\mathbf{S} \in \mathbb{R}_+^{Q \times Q}$  be the nonnegative matrix defined as

$$[\mathbf{S}]_{qr} \triangleq \begin{cases} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^{-1} \mathbf{H}_{rq}), & \text{if } r \neq q, \\ 0, & \text{otherwise.} \end{cases} \quad (63)$$

where  $\rho(\mathbf{A})$  denotes the spectral radius<sup>14</sup> of  $\mathbf{A}$ . The contraction property of the waterfilling mapping is given in the following theorem.

*Theorem 5 (Contraction property of mapping  $\mathbf{WF}$ ):*

Given  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T > \mathbf{0}$ , the mapping  $\mathbf{WF}$  defined in (61) is Lipschitz continuous on  $\mathcal{Q}$ :

$$\|\mathbf{WF}(\mathbf{Q}^{(1)}) - \mathbf{WF}(\mathbf{Q}^{(2)})\|_{F, \text{block}}^{\mathbf{w}} \leq \|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}} \times \|\mathbf{Q}^{(1)} - \mathbf{Q}^{(2)}\|_{F, \text{block}}^{\mathbf{w}}, \quad (64)$$

$\forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$ , where  $\|\cdot\|_{F, \text{block}}^{\mathbf{w}}$ ,  $\|\cdot\|_{\infty, \text{mat}}^{\mathbf{w}}$  and  $\mathbf{S}$  are defined in (62), (20), and (63), respectively, and  $\mathcal{Q} \triangleq \mathcal{Q}_1 \times \dots \times \mathcal{Q}_Q$ , with  $\mathcal{Q}_q$  given in (5). Furthermore, if

$$\|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}} < 1, \quad (65)$$

for some  $\mathbf{w} > \mathbf{0}$ , then mapping  $\mathbf{WF}$  is a block-contraction with modulus  $\alpha = \|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}}$ .

*Proof:* The proof of the theorem in the general case of arbitrary channel matrices is quite involved [21]. Here, we focus

only on the simpler case in which the direct channel matrices  $\{\mathbf{H}_{qq}\}_{q \in \Omega}$  are square and nonsingular. Under this assumption, according to Lemma 6, each component  $\mathbf{WF}(\mathbf{Q}_{-q})$  of the mapping  $\mathbf{WF}$  can be rewritten as in (58).

The proof consists in showing that the mapping  $\mathbf{WF}$  satisfies (64), with  $\alpha = \|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}}$ .

Given  $\mathbf{Q}^{(1)} = (\mathbf{Q}_q^{(1)}, \dots, \mathbf{Q}_Q^{(1)}) \in \mathcal{Q}$  and  $\mathbf{Q}^{(2)} = (\mathbf{Q}_1^{(2)}, \dots, \mathbf{Q}_Q^{(2)}) \in \mathcal{Q}$ , let define, for each  $q \in \Omega$ ,

$$e_{\mathbf{WF}_q} \triangleq \left\| \mathbf{WF}_q(\mathbf{Q}_{-q}^{(1)}) - \mathbf{WF}_q(\mathbf{Q}_{-q}^{(2)}) \right\|_F, \quad (66)$$

$$e_q \triangleq \left\| \mathbf{Q}_q^{(1)} - \mathbf{Q}_q^{(2)} \right\|_F. \quad (67)$$

Then, we have:

$$e_{\mathbf{WF}_q} = \left\| \left[ - \left( \mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(1)}) \mathbf{H}_{qq} \right)^{-1} \right]_{\mathcal{Q}_q} - \left[ - \left( \mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(2)}) \mathbf{H}_{qq} \right)^{-1} \right]_{\mathcal{Q}_q} \right\|_F \quad (68)$$

$$\leq \left\| \left( \mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(1)}) \mathbf{H}_{qq} \right)^{-1} - \left( \mathbf{H}_{qq}^H \mathbf{R}_q^{-1}(\mathbf{Q}_{-q}^{(2)}) \mathbf{H}_{qq} \right)^{-1} \right\|_F \quad (69)$$

$$= \left\| \mathbf{H}_{qq}^{-1} \left( \sum_{r \neq q} \mathbf{H}_{rq} (\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}) \mathbf{H}_{rq}^H \right) \mathbf{H}_{qq}^{-H} \right\|_F \quad (70)$$

$$\leq \sum_{r \neq q} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^{-1} \mathbf{H}_{rq}) \left\| (\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}) \right\|_F \quad (71)$$

$$\triangleq \sum_{r \neq q} [\mathbf{S}]_{qr} \left\| (\mathbf{Q}_r^{(1)} - \mathbf{Q}_r^{(2)}) \right\|_F = \sum_{r \neq q} [\mathbf{S}]_{qr} e_r, \quad (72)$$

$\forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$  and  $\forall q \in \Omega$ , where: (68) follows from (58) (Lemma 6); (69) follows from the nonexpansive property of the projector in the Frobenius norm as given in (60) (Lemma 7); (70) follows from the nonsingularity of the channel matrices  $\{\mathbf{H}_{qq}\}$ ; (71) follows from the triangle inequality [41] and from [21]

$$\left\| \mathbf{A} \mathbf{X} \mathbf{A}^H \right\|_F \leq \lambda_{\max}(\mathbf{A}^H \mathbf{A}) \|\mathbf{X}\|_F, \quad (73)$$

where  $\mathbf{X} = \mathbf{X}^H$  and  $\mathbf{A} \in \mathbb{C}^{n \times m}$ ; and (72) follows from the definition of matrix  $\mathbf{S}$  in (63).

Introducing the vectors

$$\mathbf{e}_{\mathbf{WF}} \triangleq [e_{\mathbf{WF}_1}, \dots, e_{\mathbf{WF}_Q}]^T, \quad \text{and} \quad \mathbf{e} \triangleq [e_1, \dots, e_Q]^T, \quad (74)$$

with  $e_{\mathbf{WF}_q}$  and  $e_q$  defined in (66) and (67), respectively, the set of inequalities in (72) can be rewritten in vector form as

$$\mathbf{0} \leq \mathbf{e}_{\mathbf{WF}} \leq \mathbf{S} \mathbf{e}, \quad \forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}. \quad (75)$$

Using the weighted maximum norm  $\|\cdot\|_{\infty, \text{vec}}^{\mathbf{w}}$  defined in (19) in combination with (75), we have

$$\|\mathbf{e}_{\mathbf{WF}}\|_{\infty, \text{vec}}^{\mathbf{w}} \leq \|\mathbf{S} \mathbf{e}\|_{\infty, \text{vec}}^{\mathbf{w}} \leq \|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}} \|\mathbf{e}\|_{\infty, \text{vec}}^{\mathbf{w}}, \quad (76)$$

$\forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$  and  $\mathbf{w} > \mathbf{0}$ , where  $\|\cdot\|_{\infty, \text{mat}}^{\mathbf{w}}$  is the matrix norm induced by the vector norm  $\|\cdot\|_{\infty, \text{vec}}^{\mathbf{w}}$  in (19) and defined

<sup>14</sup>The spectral radius  $\rho(\mathbf{A})$  of the matrix  $\mathbf{A}$  is defined as  $\rho(\mathbf{A}) \triangleq \max\{|\lambda| : \lambda \in \sigma(\mathbf{A})\}$ , with  $\sigma(\mathbf{A})$  denoting the spectrum of  $\mathbf{A}$  [41].

in (20) [41]. Finally, using (76) and (62), we obtain,

$$\begin{aligned} \left\| \mathbf{WF}(\mathbf{Q}^{(1)}) - \mathbf{WF}(\mathbf{Q}^{(2)}) \right\|_{F, \text{block}}^{\mathbf{w}} &= \|\mathbf{e}_{\mathbf{WF}}\|_{\infty, \text{vec}}^{\mathbf{w}} \\ &\leq \|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}} \|\mathbf{Q}^{(1)} - \mathbf{Q}^{(2)}\|_{F, \text{block}}^{\mathbf{w}}, \end{aligned} \quad (77)$$

$\forall \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)} \in \mathcal{Q}$  and  $\forall \mathbf{w} > \mathbf{0}$ , which leads to a block-contraction for the mapping  $\mathbf{WF}$  if  $\|\mathbf{S}\|_{\infty, \text{mat}}^{\mathbf{w}} < 1$ , implying condition (65). ■

## V. EXISTENCE AND UNIQUENESS OF THE NE

Using results obtained in the previous section, we can now study game  $\mathcal{G}$  and derive conditions for existence and uniqueness of the NE, as given next.

*Theorem 6: Game  $\mathcal{G}$  always admits a NE, for any set of channel matrices and transmit power of the users. Furthermore, the NE is unique if*

$$\rho(\mathbf{S}) < 1, \quad (C1)$$

where  $\mathbf{S}$  is defined in (63).

*Proof:* According to the interpretation of the waterfilling mapping  $\mathbf{WF}$  in (9) as a projector (cf. Lemma 6), the existence of a NE of game  $\mathcal{G}$  is guaranteed by the existence of a solution of the fixed-point equation (59). Invoking Theorem 1(a), the existence of a fixed-point follows from the continuity of the waterfilling projector (58) on  $\mathcal{Q}$ , for any given set of channel matrices  $\{\mathbf{H}_{rq}\}_{r,q \in \Omega}$  (implied from the continuity of the projection operator [22, Proposition 3.2c] and the continuity of each  $\mathbf{R}_{-q}^{-1}(\mathbf{Q}_{-q})$  on  $\mathcal{Q}_{-q}$ <sup>15</sup>), and from the convexity and compactness of the joint admissible strategy set  $\mathcal{Q}$ .<sup>16</sup>

According to Theorem 1(b), a sufficient condition for the uniqueness of the NE of game  $\mathcal{G}$  is that the waterfilling mapping  $\mathbf{WF}$  in (9) be a contraction with respect to some norm. It follows from Theorem 5 that  $\mathbf{WF}$  is a block-contraction if condition (65) is satisfied for some  $\mathbf{w} > \mathbf{0}$ . Since  $\mathbf{S}$  in (65) is a nonnegative matrix, there exists a positive vector  $\bar{\mathbf{w}}$  such that [22, Corollary 6.1]

$$\|\mathbf{S}\|_{\infty, \text{mat}}^{\bar{\mathbf{w}}} < 1 \Leftrightarrow \rho(\mathbf{S}) < 1, \quad (78)$$

which proves the sufficiency of (C1). ■

To give additional insight into the physical interpretation of sufficient conditions for the uniqueness of the NE, we provide the following corollary of Theorem 6.

*Corollary 1: A sufficient condition for (C1) is given by one of the two following set of conditions:*

$$\frac{1}{w_q} \sum_{r \neq q} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^H \mathbf{H}_{rq}) w_r < 1, \quad \forall q \in \Omega, \quad (C2)$$

$$\frac{1}{w_r} \sum_{q \neq r} \rho(\mathbf{H}_{rq}^H \mathbf{H}_{qq}^{-H} \mathbf{H}_{qq}^H \mathbf{H}_{rq}) w_q < 1, \quad \forall r \in \Omega, \quad (C3)$$

<sup>15</sup>This result can be proved using [37, Theorem 10.7.1].

<sup>16</sup>According to [32, Theorem 1], the existence of a NE of game  $\mathcal{G}$  can also be proved showing that the game is a concave game: 1) The set  $\mathcal{Q}_q$  of feasible strategy profiles of each player  $q$  is compact and convex; and 2) The payoff function of each player  $q$  is continuous in  $\mathbf{Q} \in \mathcal{Q}$  and concave in  $\mathbf{Q}_q \in \mathcal{Q}_q$ , for any given  $\mathbf{Q}_{-q} \in \mathcal{Q}_{-q}$  (this follows from the concavity of the log function [42]).

where  $\mathbf{w} \triangleq [w_1, \dots, w_Q]^T$  is a positive vector. □

*Remark 6 - Physical interpretation of uniqueness conditions.* Looking at conditions (C2)-(C3), it turns out, as expected, that the uniqueness of a NE is ensured if the interference among the links is sufficiently small. The importance of conditions (C2)-(C3) is that they quantify how small the interference must be to guarantee that the equilibrium is indeed unique. Specifically, condition (C2) can be interpreted as a constraint on the maximum amount of interference that each receiver can tolerate, whereas (C3) introduces an upper bound on the maximum level of interference that each transmitter is allowed to generate. This result agrees with the intuition that, as the MUI becomes negligible, the rates of the users become decoupled and then the rate-maximization problem in (4) for each user admits a unique solution.

*Remark 7 - Special cases.* Conditions in Theorem 6 and Corollary 1 for the uniqueness of the NE can be applied to arbitrary MIMO interference systems,<sup>17</sup> irrespective of the specific structure of channel matrices. Interestingly, most of the conditions known in the literature [7]-[15] for the rate-maximization game in *SISO frequency-selective* interference channels and OFDM transmission come naturally from (C1) as special cases. In fact, using the Toeplitz and circulant structure of the channel matrices  $\mathbf{H}_{rq} = \mathbf{W}\mathbf{D}_{rq}\mathbf{W}^H$  (cf. Section IV-A), matrix  $\mathbf{S}$  in the uniqueness condition (C1), defined in (63), reduces to matrix  $\mathbf{S}^{\max}$  defined in (21), showing that our uniqueness condition coincides with those given in [19], [20] and enlarges those obtained in [7]-[11] and [13], [15]. Observe that condition (C1), with  $\mathbf{S}^{\max}$  defined in (21), can be further weakened by computing the “max” over a subset of  $\{1, \dots, N\}$ , obtained from  $\{1, \dots, N\}$  by removing the subcarrier indexes where each user will never transmit, for any set of channel realizations and interference profile [18], [19]. An algorithm to compute such a set is given in [18].

Recently, in [31], the authors studied the game  $\mathcal{G}$  and proved using [32, Theorem 2.2] that the NE of the game is unique if the MUI at each receiver  $q$ , measured by the interference-to-noise ratios  $\{P_r/\sigma_q^2\}_{r \neq q}$  where  $\sigma_q^2$  is the variance of the thermal noise at receiver  $q$ , is smaller than a given *unspecified* threshold. Differently from [31], our results provide a set of sufficient conditions that can be checked in practice, since they explicitly quantify how strong the MUI must be to guarantee the uniqueness of the NE.

## VI. MIMO ASYNCHRONOUS ITERATIVE WATERFILLING ALGORITHM

According to the framework developed in Section III, to reach the Nash equilibria of game  $\mathcal{G}$ , one can use an instance of the totally asynchronous scheme of [22] (cf. Section III-B), based on the waterfilling mapping (9), called asynchronous Iterative WaterFilling Algorithm (IWFA) [21]. In the asynchronous IWFA, all the users maximize their own rate in a *totally asynchronous* way via the single user waterfilling solution (9). According to this asynchronous procedure, some users are allowed to update their strategy more frequently

<sup>17</sup>Recall that here we have concentrated on square nonsingular (direct) MIMO channels. However, condition (C1) can be generalized to the case of rectangular channel matrices [21].

than the others, and they might perform these updates using *outdated* information on the interference caused by the others. We show in the following that, whatever the asynchronous mechanism is, such a procedure converges to a stable NE of the game, under the same sufficient conditions guaranteeing the uniqueness of the equilibrium given in Theorem 6.

To provide a formal description of the proposed asynchronous IWFA, we need the following preliminary definitions. We assume, w.l.o.g., that the set of times at which one or more users update their strategies is the discrete set  $\mathcal{T} = \mathbb{N}_+ = \{0, 1, 2, \dots\}$ . Let  $\mathbf{Q}_q^{(n)}$  denote the covariance matrix of the vector signal transmitted by user  $q$  at the  $n$ -th iteration, and let  $\mathcal{T}_q \subseteq \mathcal{T}$  denote the set of times  $n$  at which  $\mathbf{Q}_q^{(n)}$  is updated (thus, at time  $n \notin \mathcal{T}_q$ ,  $\mathbf{Q}_q^{(n)}$  is left unchanged). Let  $\tau_r^q(n)$  denote the most recent time at which the interference from user  $r$  is perceived by user  $q$  at the  $n$ -th iteration (observe that  $\tau_r^q(n)$  satisfies  $0 \leq \tau_r^q(n) \leq n$ ). Hence, if user  $q$  updates his own covariance matrix at the  $n$ -th iteration, then he chooses his optimal  $\mathbf{Q}_q^{(n)}$ , according to (9), and using the interference level caused by

$$\mathbf{Q}_{-q}^{(\tau_q^q(n))} \triangleq \left( \mathbf{Q}_1^{(\tau_1^q(n))}, \dots, \mathbf{Q}_{q-1}^{(\tau_{q-1}^q(n))}, \right. \\ \left. \mathbf{Q}_{q+1}^{(\tau_{q+1}^q(n))}, \dots, \mathbf{Q}_Q^{(\tau_Q^q(n))} \right). \quad (79)$$

The overall system is said to be totally asynchronous if the following weak assumptions are satisfied for each  $q$  [22]: A1)  $0 \leq \tau_r^q(n) \leq n$ ; A2)  $\lim_{k \rightarrow \infty} \tau_r^q(n_k) = +\infty$ ; and A3)  $|\mathcal{T}_q| = \infty$ ; where  $\{n_k\}$  is a sequence of elements in  $\mathcal{T}_q$  that tends to infinity. Assumption (A1)-(A3) are standard in asynchronous convergence theory [22], and they are fulfilled in any practical implementation. In fact, (A1) simply indicates that, in the current iteration  $n$ , each user  $q$  can use only interference vectors  $\mathbf{Q}_{-q}^{(\tau_q^q(n))}$  allocated by others in previous iterations (to preserve causality). Assumption (A2) states that, for any given iteration index  $n_1$ , values of the components of  $\mathbf{Q}_{-q}^{(\tau_q^q(n))}$  in (79) generated prior to  $n_1$ , will not be used in the updates of  $\mathbf{Q}_q^{(n)}$  after a sufficiently long time  $n_2$ ; this guarantees that old information is eventually purged from the system. Finally, assumption (A3) indicates that no user fails to update his own strategy as time  $n$  goes on.

Using the above notation, the asynchronous IWFA is formally described in Algorithm 1.

---

**Algorithm 1: MIMO Asynchronous IWFA**


---

Set  $n = 0$  and  $\mathbf{Q}_q^{(0)} =$  any feasible covariance matrix;  
 for  $n = 0 : N_{it}$

$$\mathbf{Q}_q^{(n+1)} = \begin{cases} \mathbf{WF}_q \left( \mathbf{Q}_{-q}^{(\tau_q^q(n))} \right), & \text{if } n \in \mathcal{T}_q, \\ \mathbf{Q}_q^{(n)}, & \text{otherwise;} \end{cases} \quad \forall q \in \Omega \quad (80)$$

end

---

It follows directly from Theorems 2 and 5 that convergence of the algorithm is guaranteed under the following sufficient conditions.

*Theorem 7: Suppose that condition (C1) in Theorem 6 is satisfied. Then, as  $N_{it} \rightarrow \infty$ , the asynchronous IWFA,*

*described in Algorithm 1, converges to the unique NE of game  $\mathcal{G}$  for any set of feasible initial conditions and updating schedule.*  $\square$

*Remark 8 - Global convergence and robustness of the algorithm.* Even though the rate maximization game  $\mathcal{G}$  and the consequent waterfilling mapping (9) are nonlinear, condition (C1) guarantees the *global* convergence of the asynchronous IWFA. Observe that Algorithm 1 contains as special cases a plethora of algorithms, each one obtained by a possible choice of the scheduling of the users in the updating procedure (i.e., the parameters  $\{\tau_r^q(n)\}$  and  $\{\mathcal{T}_q\}$ ). Two special cases are the *sequential* and the *simultaneous* MIMO IWFA, where the users update their own strategies *sequentially* and *simultaneously*, respectively. The important result stated in Theorem 7 is that all the algorithms resulting as special cases of the asynchronous IWFA are guaranteed to reach the unique NE of the game, under the same set of convergence conditions (provided that (A1)-(A3) are satisfied), since conditions in (C1) do not depend on the particular choice of  $\{\mathcal{T}_q\}$  and  $\{\tau_r^q(n)\}$ .

*Remark 9 - Distributed nature of the algorithm.* Since the asynchronous IWFA is based on the waterfilling solution (9), it can be implemented in a distributed way, where each user, to maximize his own rate, only needs to measure the covariance matrix of the overall interference-plus-noise and waterfill over this matrix. More interestingly, according to the asynchronous scheme, the users may update their strategies using a potentially outdated version of the interference and, furthermore, some users are allowed to update their covariance matrix more often than others, without affecting the convergence of the algorithm. These features strongly relax the constraints on the synchronization of the users' updates with respect to those imposed, for example, by the simultaneous or sequential updating schemes.

*Remark 10 - Well-known cases.* The MIMO asynchronous IWFA, described in Algorithm 1 is the natural generalization of the asynchronous IWFA proposed in [20], to solve the rate-maximization game in Gaussian *SISO frequency-selective* parallel interference channels. Algorithm in [20] can be in fact obtained directly from Algorithm 1 using the following equivalences:  $\mathbf{Q}_q \Leftrightarrow \mathbf{p}_q$ ,  $\mathbf{WF}_q(\cdot) \Leftrightarrow \mathbf{wf}_q(\cdot)$ , and  $\mathcal{Q}_q \Leftrightarrow \mathcal{P}_q$ , where  $\mathbf{WF}_q(\cdot)$ ,  $\mathbf{wf}_q(\cdot)$ ,  $\mathcal{Q}_q$ , and  $\mathcal{P}_q$  are defined in (9), (16), (5), and (14), respectively. Similarly, the well-known sequential IWFA [7]-[14], [19] and simultaneous IWFA [14]-[16], [19] proposed in the literature are special cases of Algorithm 1, using the above equivalences.

## VII. NUMERICAL RESULTS

In this section, we first provide some numerical results illustrating the benefits of MIMO transceivers in the multiuser context. Then, we compare some of the proposed algorithms in terms of convergence speed.

*Example 1 - MIMO vs. SISO.* MIMO systems have shown great potential for providing high spectral efficiency in both isolated, single-user, wireless links without interference or multiple access and broadcast channels. Here we quantifies, by simulations, this potential gain for MIMO interference systems. In Figure 1, we plot the sum-rate of a two-user

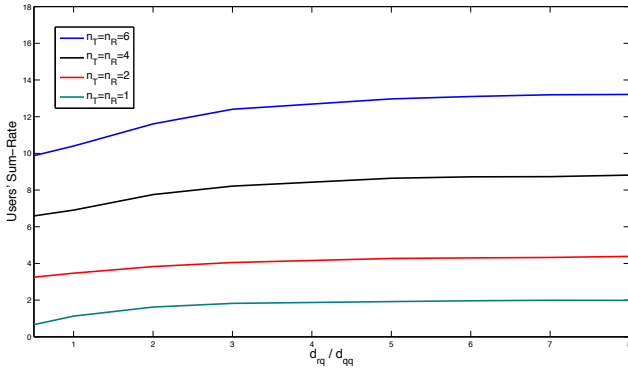


Fig. 1. Sum-Rate of the users versus  $d_{rq}/d_{qq}$ ;  $d_{rq} = d_{qr}$ ,  $d_{rr} = d_{qq} = 1$ ,  $r = 1, 2$ ,  $Q = 2$ ,  $\gamma = 2.5$ ,  $P_1/\sigma_1^2 = P_2/\sigma_2^2 = 5\text{dB}$ ,  $L_h = 6$ ,  $N = 16$ .

frequency-selective MIMO system as a function of the inter-pair distance among the links, for different number of transmit/receive antennas. The rate curves are averaged over 500 independent channel realizations, whose taps are simulated as i.i.d. Gaussian random variables with zero mean and unit variance. For the sake of simplicity, the system is assumed to be symmetric, i.e., the transmitters have the same power budget and the interference links are at the same distance (i.e.,  $d_{rq} = d_{qr}$ ,  $\forall q, r$ ), so that the cross channel gains are comparable in average sense. The path loss  $\gamma$  is assumed to be  $\gamma = 2.5$ .

From the figure one infer that, as for isolated single-user systems or multiple access/broadcast channels, also in MIMO interference channels, increasing the number of antennas at both the transmitter and the receiver side leads to a better performance. The interesting result, coming from Figure 1, is that the incremental gain due to the use of multiple transmit/receive antennas is almost independent of the interference level in the system, since the MIMO (incremental) gains in the high-interference case (small values of  $d_{rq}/d_{qq}$ ) almost coincide with the corresponding (incremental) gains obtained in the low-interference case (large values of  $d_{rq}/d_{qq}$ ), at least for the system simulated in Figure 1. This desired property is due to the fact that the MIMO channel provides more degrees of freedom for each user than those available in the SISO channel, that can be explored to find out the best partition of the available resources for each user, possibly cancelling the MUI.

*Example 2 – Sequential vs. simultaneous IWFA.* In Figure 2 we compare the performance of the sequential and simultaneous IWFA, in terms of convergence speed, for a given set of MIMO channel realizations. We consider a cellular network composed by 7 (regular) hexagonal cells, sharing the same spectrum. Hence, simultaneous transmissions of different cells can interfere with each other. The Base Stations (BS) and the Mobile Terminals (MT) are equipped with 4 antennas. For the sake of simplicity, we assume that in each cell there is only one active link, corresponding to the transmission from the BS (placed at the center of the cell) to a MT placed in a corner of the cell. According to this geometry, each MT receives an useful signal that is comparable in average sense with the

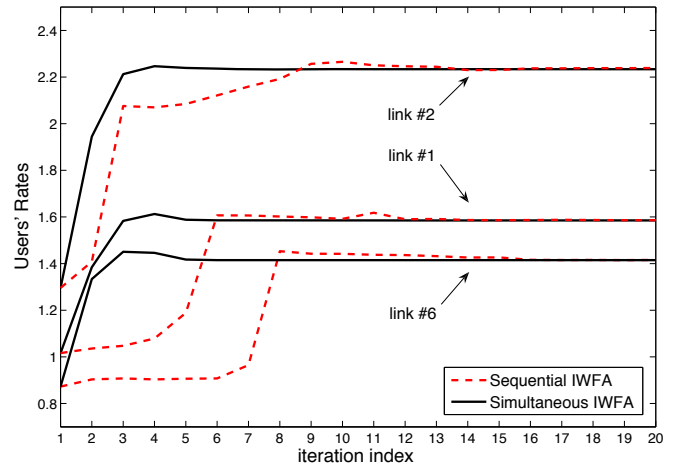


Fig. 2. Rates of the links versus iterations: sequential IWFA (dashed line curves) and simultaneous IWFA (solid line curves);  $Q = 8$ ,  $\gamma = 2.5$ ,  $P_q/\sigma_q^2 = 5\text{dB}$ ,  $\forall q \in \Omega$ ,  $L_h = 6$ ,  $N = 16$ .

interference signal transmitted by the BSs of two adjacent cells. The overall network is thus stitched out of eight  $4 \times 4$  MIMO interference wideband channels, according to (1).

In Figure 2, we show the rate evolution of the links of three cells corresponding to the sequential IWFA and simultaneous IWFA as a function of the iteration index  $n$ . To make the figure not excessively overcrowded, we plot only the curves of 3 out of 8 links. As expected, the sequential IWFA is slower than the simultaneous IWFA, especially if the number of active links  $Q$  is large, since each user is forced to wait for all the users scheduled in advance, before updating his own power allocation. The same qualitative behavior has been observed changing the channel realizations and the number of antennas.

## VIII. CONCLUSIONS

In this paper we have considered the competitive maximization of mutual information in noncooperative interfering networks in a fully distributed fashion, based on game theory. We have provided a unified view of main results obtained in the past seven years, showing that the proposed approaches, even apparently different, can be unified by our interpretation of the waterfilling solution as a proper projection onto a polyhedral set. Building on this interpretation, we have shown how to apply standard results from fixed-point and contraction theory to the rate maximization game in SISO frequency-selective channels, in order to obtain a unified set of sufficient conditions guaranteeing the uniqueness of the NE and the convergence of totally asynchronous distributed algorithms. The proposed framework has also been generalized to the (square) MIMO case. The obtained results are the natural generalization of those obtained in the SISO case.

## REFERENCES

- [1] T. Starr, J. M. Cioffi, and P. J. Silverman, *Understanding Digital Subscriber Line Technology*, Prentice Hall, NJ, 1999.

- [2] A. J. Goldsmith and S. B. Wicker, "Design Challenges for Energy-Constrained Ad Hoc Wireless Networks," *IEEE Wireless Commun. Mag.*, vol. 9, no. 4, pp. 8-27, August 2002.
- [3] I. F. Akyildiz and X. Wang, "A Survey on Wireless Mesh Networks," *IEEE Commun. Mag.*, vol. 43, no. 9, pp. 23-30, September 2005.
- [4] S. Haykin, "Cognitive Radio: Brain-Empowered Wireless Communications," *IEEE J. Select. Areas Commun.*, vol. 23, no. 2, pp. 201-220, February 2005.
- [5] M. J. Osborne and A. Rubinstein, *A Course in Game Theory*, MIT Press, 1994.
- [6] J. P. Aubin, *Mathematical Method for Game and Economic Theory*, Elsevier, Amsterdam, 1980.
- [7] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed Multiuser Power Control for Digital Subscriber Lines," *IEEE J. Select. Areas Commun.*, vol. 20, no. 5, pp. 1105-1115, June 2002.
- [8] S. T. Chung, S. J. Kim, J. Lee, and J. M. Cioffi, "A Game-theoretic Approach to Power Allocation in Frequency-selective Gaussian Interference Channels," in *Proc. 2003 IEEE International Symposium on Information Theory (ISIT 2003)*, p. 316, June 2003.
- [9] N. Yamashita and Z. Q. Luo, "A Nonlinear Complementarity Approach to Multiuser Power Control for Digital Subscriber Lines," *Optimization Methods and Software*, vol. 19, no. 5, pp. 633-652, October 2004.
- [10] G. Scutari, S. Barbarossa, and D. Ludovici, "On the Maximum Achievable Rates in Wireless Meshed Networks: Centralized versus Decentralized solutions," in *Proc. 2004 IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP-2004)*, May 2004.
- [11] G. Scutari, S. Barbarossa, and D. Ludovici, "Cooperation Diversity in Multihop Wireless Networks Using Opportunistic Driven Multiple Access," in *Proc. 2003 IEEE Workshop on Sig. Proc. Advances in Wireless Comm., (SPAWC-2003)*, pp. 170-174, June 2003.
- [12] Z.-Q. Luo and J.-S. Pang, "Analysis of Iterative Waterfilling Algorithm for Multiuser Power Control in Digital Subscriber Lines," *EURASIP J. Applied Signal Processing*, May 2006.
- [13] R. Etkin, A. Parekh, and D. Tse, "Spectrum Sharing for Unlicensed Bands," in *Proc. Allerton Conference on Communication, Control, and Computing*, Monticello, IL, September 28-30, 2005.
- [14] G. Scutari, *Competition and Cooperation in Wireless Communication Networks*, PhD. Dissertation, University of Rome, "La Sapienza", November 2004.
- [15] R. Cendrillon, J. Huang, M. Chiang and M. Moonen, "Autonomous Spectrum Balancing for Digital Subscriber Lines," *IEEE Trans. Signal Processing*, vol. 55, no. 8, p. 4241-4257, Aug. 2007.
- [16] K. W. Shum, K.-K. Leung, C. W. Sung, "Convergence of Iterative Waterfilling Algorithm for Gaussian Interference Channels," *IEEE J. Select. Areas Commun.*, vol. 25, no. 6, pp. 1091-1100, Aug. 2007.
- [17] G. Scutari, D. P. Palomar, and Sergio Barbarossa, "Distributed Totally Asynchronous Iterative Waterfilling for Wideband Interference Channel with Time/Frequency Offset," in *Proc. of the IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP)*, Honolulu, Hawaii, USA, April 15-20, 2007.
- [18] G. Scutari, D. P. Palomar, and S. Barbarossa, "Optimal Linear Precoding Strategies for Wideband Non-Cooperative Systems based on Game Theory-Part I: Nash Equilibria," *IEEE Trans. Signal Processing*, Vol. 56, no. 3, pp. 1230-1249, March 2008.
- [19] G. Scutari, D. P. Palomar, and S. Barbarossa, "Optimal Linear Precoding Strategies for Wideband Non-Cooperative Systems based on Game Theory-Part II: Algorithms," *IEEE Trans. on Signal Processing*, Vol. 56, no. 3, pp. 1250-1267, March 2008. See also *Proc. of IEEE International Symposium on Information Theory (ISIT)*, Seattle, WA, USA, July 9-14, 2006.
- [20] G. Scutari, D. P. Palomar, and S. Barbarossa, "Asynchronous Iterative Waterfilling for Gaussian Frequency-Selective Interference Channels," *IEEE Trans. on Information Theory*, vol. 54, no. 7, pp. 2868-2878, July 2008. See also *Proc. IEEE Workshop on Signal Proc. Advances in Wireless Comm. (SPAWC 2006)*, Cannes, France, July 2-5, 2006; and *Proc. of IEEE Information Theory and Applications (ITA) Workshop*, San Diego, CA, USA, Jan. 29 - Feb. 2, 2007.
- [21] G. Scutari, D. P. Palomar, and S. Barbarossa, "The MIMO Iterative Waterfilling Algorithm," submitted to *IEEE Trans. on Signal Processing*. See also *Proc. of the IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP)*, Las Vegas, USA, March 30 - April 4, 2008.
- [22] D. P Bertsekas and J.N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*, Athena Scientific, 2nd Ed., 1989.
- [23] R. D. Yates, "A Framework for Uplink Power Control in Cellular Radio Systems," *IEEE J. Select. Areas Commun.*, vol. 13, no 7, pp. 1341-1347, September 1995.
- [24] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, SIAM Ed., 2000.
- [25] R. P. Agarwal, M. Meehan, and D. O' Regan, *Fixed Point Theory and Application*, Cambridge University Press, 2001.
- [26] S. N. Elaydi, *An Introduction to Difference Equations*, Springer, 3rd ed., 2005.
- [27] E. Larsson and E. Jorswieck, "Competition and Collaboration on the MISO Interference Channel," in *Proc. of Allerton Conference on Communication, Control, and Computing, 2007* (invited paper).
- [28] S. Ye and R. S. Blum, "Optimized Signaling for MIMO Interference Systems With Feedback," *IEEE Trans. Signal Processing*, vol. 51, no. 11, pp. 2839-2848, November 2003.
- [29] M. F. Demirkol and M. A. Ingram, "Power-Controlled Capacity for Interfering MIMO Links," in *Proc. IEEE Vehicular Technology Conference (VTC 2001)*, 2001.
- [30] C. Liang and K. R. Dandekar, "Power Management in MIMO Ad Hoc Networks: A Game-Theoretic Approach," *IEEE Trans. Wireless Commun.*, vol. 6, no. 4, pp. 2866-2882, April 2007.
- [31] G. Arslan, M. Fatih Demirkol and Y. Song, "Equilibrium efficiency improvement in MIMO interference systems: a decentralized stream control approach," *IEEE Transaction Wireless Commun.*, vol. 6, no. 8, pp. 2984-2993, August 2007.
- [32] J. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave n-Person Games," *Econometrica*, vol. 33, no. 3, pp. 520-534, July 1965.
- [33] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, 1991.
- [34] J. G. David Forney and M. V. Eyuboglu, "Combined Equalization and Coding Using Precoding," *IEEE Commun. Mag.*, vol. 29, no. 12, pp. 25-34, Dec. 1991.
- [35] H. K. Khalil, *Nonlinear Systems, Prentice Hall, Third Ed., 2002*.
- [36] D. Liberzon, *Switching in Systems and Control*, Springer, 1993.
- [37] S. L. Campbell and C.D. Meyer, *Generalized Inverse of Linear Transformations*, Dover Publications, 1991.
- [38] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press, February 22, 2005.
- [39] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Academic Press (Cambridge 1992).
- [40] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer, 2000.
- [41] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1985.
- [42] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2003.



**Gesualdo Scutari** Gesualdo Scutari (S'05-M'06) received the electrical engineering (with honors) and Ph.D. degrees from University of Rome "La Sapienza," Rome, Italy, in 2001 and 2004, respectively. During 2003, he held a visiting research appointment at the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley; during 2007 and 2008, he was a post-doctoral fellow at the Department of Electronic and Computer Engineering at the Hong Kong University of Science and Technology, Hong Kong. He currently holds a post-doc position at INFOCOM Department, University of Rome "La Sapienza". He has participated in two European projects on multiantenna systems and multihop systems (IST SATURN and IST-ROMANTIK). He is currently involved in the European project WINSOC, on wireless sensor networks, and in the European project SURFACE, on reconfigurable air interfaces for wideband multiantenna communication systems. His primary research interests include communication aspects of wireless multiple-input multiple-output (MIMO) channels, with special emphasis on convex optimization theory and game theory applied to communications systems; sensor networks; and distributed decisions. Dr. Scutari received the 2006 Best Student Paper Award at ICASSP'06.



**Daniel P. Palomar** Daniel P. Palomar (S'99-M'03) received the Electrical Engineering and Ph.D. degrees (both with honors) from the Technical University of Catalonia (UPC), Barcelona, Spain, in 1998 and 2003, respectively. He is an Assistant Professor in the Department of Electronic and Computer Engineering at the Hong Kong University of Science and Technology, Hong Kong. He has held several research appointments, namely, at King's College London (KCL), London, UK, during 1998; Technical University of Catalonia (UPC), Barcelona,

from January 1999 to December 2003; Stanford University, Stanford, CA, from April to November 2001; Telecommunications Technological Center of Catalonia (CTTC), Barcelona, from January to December 2002; Royal Institute of Technology (KTH), Stockholm, Sweden, from August to November 2003; University of Rome "La Sapienza", Rome, Italy, from November 2003 to February 2004; Princeton University, Princeton, NJ, from March 2004 to July 2006. His primary research interests include information-theoretic and signal processing aspects of MIMO channels, with special emphasis on convex optimization theory and majorization theory applied to communication systems. He is an Associate Editor of *IEEE Trans. on Signal Processing*, a guest editor of the *IEEE Journal on Selected Areas in Communications (JSAC)* 2008 special issue on "Game Theory in Communication Systems", and the lead guest editor of the *IEEE JSAC* 2007 special issue on Optimization of MIMO Transceivers for Realistic Communication Networks. Dr. Palomar received a 2004/06 Fulbright Research Fellowship; the 2004 Young Author Best Paper Award by the IEEE Signal Processing Society; (co-recipient of) the 2006 Best Student Paper Award at ICASSP'06; the 2002/03 best Ph.D. prize in Information Technologies and Communications by the Technical University of Catalonia (UPC); the 2002/03 Rosina Ribalta first prize for the Best Doctoral Thesis in Information Technologies and Communications by the Epson Foundation; and the 2004 prize for the best Doctoral Thesis in Advanced Mobile Communications by the Vodafone Foundation and COIT.



**Sergio Barbarossa** Sergio Barbarossa (S'82, M'88) received the M.Sc. in 1984 and the Ph.D. in electrical engineering in 1988, both from the University of Rome {"La Sapienza", Rome, Italy. He joined the Radar System Division of Selenia in 1985 as a Radar System Designer. In 1987, he was a research engineer at the Environmental Research Institute of Michigan (ERIM), Ann Arbor, Michigan. From 1988 until 1991, he was an assistant professor at the University of Perugia. In November 1991, he joined the University of Rome La Sapienza, where he is

now a full professor and director of graduate studies. He has held positions as visiting scientist and visiting professor at the University of Virginia (1995 and 1997), the University of Minnesota (1999), and the Polytechnic University of Catalonia (2001). He has participated in international projects on synthetic aperture radar systems, flown on the space shuttle, and he has been the primary investigator in four European projects on multiantenna systems and multihop systems. He is currently the scientific responsible of the European project WINSOC (on wireless sensor networks), and he is one of the primary investigators of the European project SURFACE (on reconfigurable air interfaces for wideband multiantenna communication systems). From 1998 to 2004, he was a member of the IEEE Signal Processing for Communications Technical Committee. He served as an associate editor of the *IEEE Transactions on Signal Processing* from 1998 to 2001 and from 2004 to 2006. He has co-edited a special issue of the *EURASIP Journal of Applied Signal Processing* on MIMO Communications and Signal Processing and a special issue of the *IEEE Journal on Selected Areas in Communications* on Optimization of MIMO Transceivers for Realistic Communication Networks: Challenges and Opportunities. He received the 2000 IEEE Best Paper Award from the IEEE Signal Processing Society for a co-authored paper in the area of signal processing for communications. He is the author of a research monograph titled {"Multiantenna Wireless Communication Systems." His current research interests lie in the areas of sensor networks, cooperative communications, and distributed decision.