A Dual Perspective on Separable Semidefinite Programming With Applications to Optimal Downlink Beamforming

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***Abstract*—This paper considers the downlink beamforming optimization problem that minimizes the total transmission power subject to global shaping constraints and individual shaping constraints, in addition to the constraints of quality of service (QoS) measured by signal-to-interference-plus-noise ratio (SINR). This beamforming problem is a separable homogeneous quadratically constrained quadratic program (QCQP), which is difficult to solve in general. Herein we propose efficient algorithms for the problem consisting of two main steps: 1) solving the semidefinite programming (SDP) relaxed problem, and 2) formulating a linear program (LP) and solving the LP (with closed-form solution) to find a rank-one optimal solution of the SDP relaxation. Accordingly, the corresponding optimal beamforming problem (OBP) is proven to be "hidden" convex, namely, strong duality holds true under certain mild conditions. In contrast to the existing algorithms based on either the rank reduction steps (the purification process) or the Perron-Frobenius theorem, the proposed algorithms are based on the linear program strong duality theorem.**

***Index Terms*—Downlink beamforming, LP approach, rank-constrained solution, SDP relaxation.**

I. INTRODUCTION

DOWNLINK transmit beamforming has recently received a lot of attention, since techniques of beamforming can be utilized to achieve higher spectrum efficiency and larger downlink capacity for a communication system by equipping the base stations with antenna arrays (see [1]). The base stations of the system transmit the weighted signals to all intended co-channel users simultaneously, and the beamforming vectors (the weights) are jointly designed, one per user, in the optimization problem.

A basic formulation for the optimal downlink beamforming problem is to minimize the transmission power subject to the individual quality of service (QoS) constraint of each user as well as some additional beam pattern constraints. The QoS is often measured in terms of signal-to-interference-plus-noise ratio (SINR). In the seminal work [2] and [3], the optimization problem with SINR constraints was solved resorting to semidefinite program (SDP) relaxation technique and the Perron-Frobenius theory for matrices with nonnegative entries. In [4], the authors further considered the optimal beamforming problem (OBP) additionally with indefinite shaping constraints (individual shaping constraint termed herein). In [5], we applied rank reduction techniques to yield an optimum of the SDP relaxation of the problem with SINR constraints as well as either two soft-shaping interference constraints or two groups of individual shaping constraints (i.e., two indefinite shaping constraints on each user). There are also other early works on the optimization with QoS constraints, for instance, [6] and [7], where the authors solved an equivalent virtual uplink formulation of the optimization problem. For multicast beamforming, we refer to the overview paper [1], the book chapter [8], and the earlier paper [9].

An alternative formulation for the optimal downlink beamforming problem is to maximize the minimum SINR value among the intended receivers subject to the power budget constraint. The resulting optimization problem and the relationship between the two formulations have been studied in [7] and [4] for the unicast beamforming case, and in [10] and [11] for the multicast beamforming case. In addition to the above transmit beamforming problems, one may refer to the paper [1] for optimization problems of receive beamforming and network beamforming.

The beamforming problem we study herein is the power minimization problem subject to the QoS constraints, global shaping constraints, and individual shaping constraints. Particularly, it is of interest to introduce the soft-shaping interference constraints which belong to a subclass of the global shaping constraints. Their introduction is motivated as a way to protect coexisting wireless systems which may operate in the same spectral band, located in the same area. In words, when optimizing the beamforming vectors, we take into account that the interference level generated from the system of interest to the users of coexisting systems should be under a very low level. The individual shaping constraints are introduced to limit the beam pattern for each individual user, and the motivation has been addressed in [4]. The problem belongs to a class of the nonconvex separable quadratically constrained quadratic program (QCQP), which is known to be NP-hard in general (e.g., see [8] and [12]). In particular, a convex relaxation of the optimization problem may or may not be tight; for instance, the SDP relaxation of the problem could have optimal solutions

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of rank higher than one only, or equal to one as well as higher than one. Nonetheless, it is possible for us either to find some instances of the nonconvex QCQP that can still be solved efficiently, or to find a polynomial approximation algorithm for some instances of it with a provable and satisfactory approximation performance guarantee (e.g., see [12]).

In this paper, we aim at establishing another efficient algorithm for the OBP, by which we enlarge the class of "solvable" instances of the downlink beamforming optimization problem. The presented algorithms are based on solving the SDP relaxation of the beamforming optimization problem and then solving a formulated linear program (LP) to retrieve a rank-one solution of the SDP (i.e., an optimal solution of the original beamforming problem), based on the linear program strong duality. Specially, the two main contributions of the paper are: i) simplified algorithms providing a rank-one solution, as well as a rank-constrained solution with a prefixed rank profile, for a separable SDP; ii) two more solvable subclasses of the OBP identified under some mild conditions (besides those subclasses of the problem discussed in [5]).

In contrast to the iterative procedure of rank reduction (also known as purification process) for the separable SDPs in [5], the presented algorithms herein output a rank-one solution by solving an LP in one single step, thus the implementation is easier by a standard solver since the algorithms merely involve solving an SDP and an LP (which in fact has a closed-form solution). A limitation of [5] is that the rank reduction procedure can be employed to solve the OBP with up to two soft-shaping interference constraints, while the algorithms herein can be applicable to the problem with no soft-shaping constraints, but with multiple groups of the individual shaping constraints. Another major difference is that the rank reduction procedure works on the primal optimal solution set only, i.e., searching primal optimal solutions of lower rank, while the algorithms of this paper capitalize on some properties of the dual optimal solution to formulate LPs and thus a rank-one solution of the primal SDP is generated through the dual.

The outline of the paper is as follows. Section II gives the system model and formulates the optimal downlink beamforming problem. The individual shaping constraint and global shaping constraint are introduced and discussed. In Section III, we revisit and extend the rank reduction process for separable SDP from the primal perspective as in [5], while in Section IV we propose algorithms for separable SDP from the dual perspective. In Section V, the algorithms are generalized to cope with more beam pattern constraints. In Section VI, we summarize particular instances of solvable of the optimal (unicast) downlink beamforming problem. In Section VII, we present some numerical results for simulated scenarios of the OBP. Finally, Section VII draws some conclusions.

Notation: We adopt the notation of using boldface for vectors $\boldsymbol{a}$ (lower case), and matrices $\boldsymbol{A}$ (upper case). The transpose operator and the conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$ respectively. $\mathrm{tr}(\cdot)$ is the trace of the square matrix argument, $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively the identity matrix and the matrix with zero entries (their size is determined from the context). The letter j represents the imaginary unit (i.e., $j=\sqrt{-1}$), while the letter i often serves as index in this paper. For any complex number x, we use $\Re(x)$ and $\Im(x)$ to denote respectively the real and the imaginary part of x, $|x|$ and $\arg(x)$ to represent respectively the modulus and the argument of x, and x^* to stand for the conjugate of x. The Euclidean norm of the vector $\boldsymbol{x}$ is denoted by $\|\boldsymbol{x}\|$. $\lambda_{\max}(\boldsymbol{A})$ stands for the largest eigenvalue of $\boldsymbol{A}$. We employ $\boldsymbol{A}\bullet\boldsymbol{B}$ standing for the inner product $\mathrm{tr}(\boldsymbol{AB})$ of Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The curled inequality symbol $\succeq$ (and its strict form $\succ$) is used to denote generalized inequality: $\boldsymbol{A}\succeq\boldsymbol{B}$ means that $\boldsymbol{A}-\boldsymbol{B}$ is an Hermitian positive semidefinite matrix ($\boldsymbol{A}\succ\boldsymbol{B}$ for positive definiteness). We denote by $\mathbb{R}_+^L$ the set of L-dimension nonnegative vectors.

II. Problem Formulation of Optimal Downlink Beamforming Problem

A. System Model

Consider a wireless system where N base stations (BSs), each with an array of K antenna elements, serve L single-antenna users over a common frequency band. Each user m is assigned to base station $\kappa_m\in\{1,\ldots,N\}$ and receives an independent data stream $s_m(t)$ from the base station. It is assumed that the scalar-valued data streams $s_m(t)$, $m=1,\ldots,L$, are temporally white with zero mean and unit variance. The transmitted signal by the kth base station is $\boldsymbol{x}_k(t)=\sum_{m\in\mathcal{I}_k}\boldsymbol{w}_m s_m(t)$, where $\boldsymbol{w}_m\in\mathbb{C}^K$ is the transmit beamforming vector for user m, and the index set $\mathcal{I}_k\subseteq\{1,\ldots,L\}$ represents the set of users assigned to base station k.

The signal received by user m is expressed with the baseband signal model

$$r_m(t)=\boldsymbol{h}_{m,\kappa_m}^H\boldsymbol{x}_{\kappa_m}(t)+\sum_{k=1,\ k\neq\kappa_m}^{N}\boldsymbol{h}_{m,k}^H\boldsymbol{x}_k(t)+n_m(t), \quad (1)$$

where $\boldsymbol{h}_{m,k}\in\mathbb{C}^K$ is the channel vector between base station k and user m, and $n_m(t)$ is a zero-mean complex Gaussian noise with variance σ_m^2. The SINR of user m is given by

$$\mathrm{SINR}_m=\frac{\boldsymbol{w}_m^H\boldsymbol{R}_{mm}\boldsymbol{w}_m}{\sum_{l=1,\,l\neq m}^{L}\boldsymbol{w}_l^H\boldsymbol{R}_{ml}\boldsymbol{w}_l+\sigma_m^2}, \quad (2)$$

where $\boldsymbol{R}_{ml}=\mathrm{E}[\boldsymbol{h}_{m,\kappa_l}\boldsymbol{h}_{m,\kappa_l}^H]$ is the downlink channel correlation matrix. Note that (2) defines the *average* SINR, and it is a long-term SINR (as opposed to the instantaneous SINR). Another application where an expression similar to (2) arises is in a multiple-input multiple-output (MIMO) communication system using orthogonal space-time block codes (OSTBC) in combination with beamforming (cf. [13]); in particular, each term of the form $\boldsymbol{w}_l^H\boldsymbol{R}_{ml}\boldsymbol{w}_l$ in (2), $l=1,\ldots,L$, respectively, becomes $\mathrm{tr}(\boldsymbol{W}_l\boldsymbol{R}_{ml}\boldsymbol{W}_l)$, where $\boldsymbol{W}_l=[\boldsymbol{w}_{l1}\ \cdots\ \boldsymbol{w}_{lK}]$.

B. Beam Pattern Constraints

In the classical optimal downlink beamforming problem (for instance, see [2] and [3]), the beamforming vectors are designed to ensure that each user can retrieve the signal of interest with the desired QoS, which is usually described by the SINR constraint $\mathrm{SINR}_m\geq\rho_m$ with a prefixed threshold ρ_m for user m.

Besides, some additional constraints on the beamforming vectors described next may be of interest in a modern wireless communication system.

1) Individual Shaping Constraints: In this paper, we consider the following I groups of individual shaping constraints on the beamforming vectors $\boldsymbol{w}_l$, $l=1,\dots,L$ (cf. [4])

$$\boldsymbol{w}_l^H \boldsymbol{D}_{il} \boldsymbol{w}_l = (\geq) 0,\ \forall l \in (\notin) \mathcal{E}_i,\ i=1,\dots,I \tag{3}$$

where $\mathcal{E}_i$, $i=1,\dots,I$, are subsets of the index set $\{1,\dots,L\}$ of users within the system, and the parameters $\boldsymbol{D}_{il}$, $i=1,\dots,I$, $l=1,\dots,L$, are Hermitian matrices, i.e., $\boldsymbol{D}_{il}$ may have negative and/or nonnegative eigenvalues.

Note that the constraints (3) affect each user individually, in other words, the desired beamforming vector for user l is limited by

$$\boldsymbol{w}_l^H \boldsymbol{D}_{il} \boldsymbol{w}_l \trianglerighteq_{il} 0,\ i=1,\dots,I$$

where $\trianglerighteq_{il} \in \{\geq, =\}$. By properly selecting $\boldsymbol{D}_{il}$, one can formulate different kinds of constraints on the beamforming vectors (for example, see [4, Sec. V] for the discussion of various applications for individual shaping constraints).

2) Global Shaping Constraints: Herein, the following global shaping constraints (compare with the individual constraints) on beamforming vectors are considered:

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{T}_{ml} \boldsymbol{w}_l = \sum_{l=1}^{L} \boldsymbol{T}_{ml} \bullet \boldsymbol{w}_l \boldsymbol{w}_l^H \leq \tau_m, \quad m=1,\dots,M \tag{4}$$

where matrices $\boldsymbol{T}_{ml}$ are Hermitian. In general, every $\boldsymbol{T}_{ml}$ could be any Hermitian matrix and we will specify $\boldsymbol{T}_{ml}$ in the later applications. We now consider some specific constraints that belong to the global constraints.

Soft-Shaping Interference Constraints. In some scenarios, it is necessary to limit the amount of co-channel interference generated along some particular directions, e.g., to protect coexisting systems (see [14]), defined as follows. Let $\boldsymbol{h}_{m,k}$ be the channel between base station k and coexisting system's user m, $m \in \{L+1,\dots,M\}$, where we reserve the indexes $\{1,\dots,L\}$ for users within the system for which beamforming vectors are designed. The amount of interference received by coexisting system's user m from the system is

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H (\boldsymbol{h}_{m,\kappa_l} \boldsymbol{h}_{m,\kappa_l}^H) \boldsymbol{w}_l, \quad m=L+1,\dots,M.$$

Let us adopt the notation

$$\boldsymbol{S}_{ml} = \boldsymbol{h}_{m,\kappa_l} \boldsymbol{h}_{m,\kappa_l}^H,\ m=L+1,\dots,M,\ l=1,\dots,L. \tag{5}$$

The soft-shaping constraint limiting the amount of interference received to a given tolerant value τ_m is

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, \quad m=L+1,\dots,M. \tag{6}$$

Null-Shaping Interference Constraints. By setting $\tau_m = 0$ in (6), we guarantee no interference generated at that location; this type of constraint is termed null-shaping interference constraint or, in short, null interference constraint (see [14]). It can be verified that a null-shaping interference constraint is mathematically equivalent to a group of individual equality shaping constraints, that is

$$\sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{il} \boldsymbol{w}_l \leq 0$$

where $\boldsymbol{S}_{il} \succeq \boldsymbol{0}$, $\forall l$, is equivalent to $\boldsymbol{w}_l^H \boldsymbol{S}_{il} \boldsymbol{w}_l = 0,\ \forall l$.

Robust Soft-Shaping/Null-Shaping Interference Constraints. Suppose that external user m is located at θ_{mk} relative to the array broadside of base station k. Let the channel between base station k and external user m be given by

$$\boldsymbol{h}(\theta_{mk}) = [1\ e^{j\phi_{mk}}\ \cdots\ e^{j(K-1)\phi_{mk}}]^T, \quad k=1,\dots,N, \tag{7}$$

where $\phi_{mk} = 2\pi d \sin(\theta_{mk})/\lambda$, d is the antenna element separation, and λ is the carrier wavelength. Note that (7) defines a Vandermonde channel vector which arises when a uniform linear antenna array (ULA) is used at the transmitter under far field, line-of-sight propagation conditions. The interference power received by external user m from base stations k, $k=1,\dots,N$, is

$$\sum_{k=1}^{N} \boldsymbol{h}(\theta_{mk})^H \boldsymbol{W}_k \boldsymbol{h}(\theta_{mk})$$

where $\boldsymbol{W}_k = \sum_{l \in \mathcal{I}_k} \boldsymbol{w}_l \boldsymbol{w}_l^H$. To keep the interference under threshold value τ_m in a small region $\Delta\theta$ about θ_{mk} ($\forall k$), we may impose the robust soft-shaping interference constraint

$$\sum_{k=1}^{N} \boldsymbol{h}(\theta_{mk}+\Delta\theta)^H \boldsymbol{W}_k \boldsymbol{h}(\theta_{mk}+\Delta\theta) \leq \tau_m, \text{ for small } \Delta\theta. \tag{8}$$

When $\Delta\theta$ is small enough, the power received $\boldsymbol{h}(\theta_{mk}+\Delta\theta)^H \boldsymbol{W}_k \boldsymbol{h}(\theta_{mk}+\Delta\theta)$ can be approximated using the first-order Taylor expansion of the channel (cf. [15]) as $\boldsymbol{h}(\theta_{mk}+\Delta\theta)^H \boldsymbol{W}_k \boldsymbol{h}(\theta_{mk}+\Delta\theta) \simeq (\boldsymbol{h}(\theta_{mk}) + \Delta\theta g(\theta_{mk}))^H \boldsymbol{W}_k (\boldsymbol{h}(\theta_{mk}) + \Delta\theta g(\theta_{mk}))$, where $\boldsymbol{g}(\theta) = d\boldsymbol{h}(\theta)/d\theta$. Therefore, a parametric way to approximately ensure (8) is to set the response power along the derivative of the channel to zero:

$$\sum_{k=1}^{N} \boldsymbol{g}(\theta_{mk})^H \boldsymbol{W}_k \boldsymbol{g}(\theta_{mk}) = 0, \tag{9}$$

on top of the nominal soft-shaping interference constraint

$$\sum_{k=1}^{N} \boldsymbol{h}(\theta_{mk})^H \boldsymbol{W}_k \boldsymbol{h}(\theta_{mk}) \leq \tau_m. \tag{10}$$

C. Optimal Downlink Beamforming Problem and SDP Relaxation

This paper focuses on the design of downlink beamforming vectors $\boldsymbol{w}_l, l = 1, \ldots, L$, that minimize the total transmit power at the base stations while ensuring a desired QoS for each user, as well as global shaping and individual shaping constraints. Specifically, we consider the beamforming optimization problem (OBP) shown in (11) at the bottom of the page, which can be rewritten equivalently into a separable homogeneous QCQP (see [8]) as (12) shown at the bottom of the page. In [5], we considered problem (OBP) with two groups of individual shaping constraints only, while more groups of individual shaping constraints are involved herein (a total of I groups). Clearly, the problem is a nonconvex separable homogeneous QCQP problem, which is known to be NP-hard in general (see [8]), and its SDP relaxation may not be tight. Nevertheless, there are some instances of separable QCQP (OBP) having strong duality (see [2]–[4] and [5]). The SDP relaxation of (12) is (SDR) shown in (13) at the bottom of the page. It is known that an SDP is convex and that a general-rank solution of it can be obtained by interior-point methods in polynomial time (see [16, Ch. 4] for instance) provided it is solvable[1]. Also there are several easy-to-use solvers for SDPs. We highlight that solving (OBP) amounts to finding a rank-one optimal solution of its SDP relaxation problem (SDR); however, retrieving a rank-constrained (e.g., rank-one) solution[2] from a solution of arbitrary rank is often nontrivial (if possible at all).

In this paper, we will build another approach for the OBP. The presented approach simply consists of solving an SDP and an LP. This approach gives a rank-one solution and, more generally a rank-constrained solution with a prefixed rank profile. We will summarize all the particular instances of the general QCQP downlink beamforming problem (OBP) described by a table in the end of this paper (i.e., Table I in Section VI).

III. Separable Semidefinite Programming From the Primal: Revisit and Extension

Consider a separable SDP as follows:

$$(\text{P0})\begin{cases}\underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m, \quad m = 1, \ldots, M, \\ & \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l = (\geq) 0, \ \forall l \in (\notin) \mathcal{E}_i, \quad i = 1, \ldots, I, \\ & \boldsymbol{X}_l \succeq \boldsymbol{0}, \quad l = 1, \ldots, L \end{cases} \tag{14}$$

where $\boldsymbol{C}_l, \ \boldsymbol{A}_{ml} \in \mathcal{H}^K, l = 1, \ldots, L, m = 1, \ldots, M$, i.e., they are Hermitian matrices (not necessarily positive semidefinite), $b_m \in \mathbb{R}, \trianglerighteq_m \in \{\geq, =, \leq\}, m = 1, \ldots, M$. The dual problem

[1]By "solvable" we mean that the problem is feasible, bounded, and the optimal value is attained (e.g., see [16, p. 13]).

[2]A rank-one solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ of (SDR) means $\text{rank}(\boldsymbol{X}_l^\star) = 1, \forall l$.

$$(\text{OBP})\begin{cases}\underset{\boldsymbol{w}_l \in \mathbb{C}^K, \ l=1,\ldots,L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{w}_l \\ \text{subject to} & \dfrac{\boldsymbol{w}_m^H \boldsymbol{R}_{mm} \boldsymbol{w}_m}{\sum_{l \neq m} \boldsymbol{w}_l^H \boldsymbol{R}_{ml} \boldsymbol{w}_l + \sigma_m^2} \geq \rho_m, \quad m = 1, \ldots, L, \\ & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, \quad m = L+1, \ldots, M, \\ & \boldsymbol{w}_l^H \boldsymbol{D}_{il} \boldsymbol{w}_l = (\geq) 0, \ \forall l \in (\notin) \mathcal{E}_i, \quad i = 1, \ldots, I. \end{cases} \tag{11}$$

$$\begin{cases}\underset{\boldsymbol{w}_l \in \mathbb{C}^K, \ l=1,\ldots,L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{w}_l \\ \text{subject to} & \boldsymbol{w}_m^H \boldsymbol{R}_{mm} \boldsymbol{w}_m - \rho_m \sum_{l \neq m} \boldsymbol{w}_l^H \boldsymbol{R}_{ml} \boldsymbol{w}_l \geq \rho_m \sigma_m^2, \quad m = 1, \ldots, L, \\ & \sum_{l=1}^{L} \boldsymbol{w}_l^H \boldsymbol{S}_{ml} \boldsymbol{w}_l \leq \tau_m, \quad m = L+1, \ldots, M, \\ & \boldsymbol{w}_l^H \boldsymbol{D}_{il} \boldsymbol{w}_l = (\geq) 0, \ \forall l \in (\notin) \mathcal{E}_i, \quad i = 1, \ldots, I. \end{cases} \tag{12}$$

$$(\text{SDR})\begin{cases}\underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{I} \bullet \boldsymbol{X}_l \\ \text{subject to} & \boldsymbol{R}_{mm} \bullet \boldsymbol{X}_m - \rho_m \sum_{l \neq m} \boldsymbol{R}_{ml} \bullet \boldsymbol{X}_l \geq \rho_m \sigma_m^2, \quad m = 1, \ldots, L, \\ & \sum_{l=1}^{L} \boldsymbol{S}_{ml} \bullet \boldsymbol{X}_l \leq \tau_m, \quad m = L+1, \ldots, M, \\ & \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l = (\geq) 0, \ \forall l \in (\notin) \mathcal{E}_i, \quad i = 1, \ldots, I, \\ & \boldsymbol{X}_l \succeq \boldsymbol{0}, \quad l = 1, \ldots, L. \end{cases} \tag{13}$$

of (P0) is (15) shown at the bottom of the page, where $\trianglerighteq_m^*$ is defined according to

$$\trianglerighteq_m^* \text{ is } \begin{cases} \geq, & \text{if } \trianglerighteq_m \text{ is } \geq \\ \text{unrestricted}, & \text{if } \trianglerighteq_m \text{ is } =, \\ \leq, & \text{if } \trianglerighteq_m \text{ is } \leq \end{cases} \quad m = 1, \ldots, M. \tag{16}$$

In [5], we aimed at generating a rank-constrained solution of (P0) from the primal perspective, which means that we update the primal optimal solutions of the SDP by a rank reduction procedure while fixing a dual optimal solution. In this section, we shall elaborate how to get a rank-constrained solution for problem (P0) with the additional I groups of individual semidefinite shaping constraints as well as the first M (global) constraints, from the primal perspective. This section serves the purpose of a revisit and slight extension of [5], and not a major part of this paper.

A. Revisit and Extension

Suppose that the parameters $\boldsymbol{D}_{il}$ in problem (P0) comply with

$$\boldsymbol{D}_{il} \succeq \mathbf{0}, \text{ or } \boldsymbol{D}_{il} \preceq \mathbf{0}, \ \forall i, l \tag{17}$$

and that (P0) and (D0) are solvable. Let $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ and $(y_1^\star, \ldots, y_M^\star, z_{11}^\star, \ldots, z_{IL}^\star, \boldsymbol{Z}_1^\star, \ldots, \boldsymbol{Z}_L^\star)$ be optimal solutions of problems (P0) and (D0), respectively. Then, they satisfy the complementary conditions (see [16, Th. 1.7.1, 4)] for instance) of SDPs (P0) and (D0):

$$\boldsymbol{X}_l^\star \bullet \boldsymbol{Z}_l^\star = 0, \ l = 1, \ldots, L \tag{18}$$

$$y_m^\star \left(\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star - b_m \right) = 0, \ m = 1, \ldots, M \tag{19}$$

$$z_{il}^\star (\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l^\star) = 0, \ \forall l, \ i = 1, \ldots, I \tag{20}$$

where (18) is equivalent to $\boldsymbol{X}_l^\star \boldsymbol{Z}_l^\star = \mathbf{0}$, $l = 1, \ldots, L$, since $\boldsymbol{X}_l^\star \succeq \mathbf{0}$ and $\boldsymbol{Z}_l^\star \succeq \mathbf{0}$, $\forall l$. Similar to the purification process introduced in Theorem 3.2 and Algorithm 1 of [5], a rank-constrained optimal solution of (P0) can be constructed from a general-rank solution, and we have the theorem and algorithm for (P0).

Theorem 3.1: Suppose that the parameters $\boldsymbol{D}_{il}$ comply with (17). Suppose that the separable SDP (P0) and its dual (D0) are solvable. Then, problem (P0) has always an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ such that

$$\sum_{l=1}^{L} \text{rank}^2(\boldsymbol{X}_l^\star) \leq M. \tag{21}$$

Proof: See Appendix A. ■

As seen from the proof, Algorithm 1 summarizes the rank reduction procedure.

Algorithm 1: Rank Reduction Procedure for Separable SDP

`Input` : $\boldsymbol{C}_l, \boldsymbol{A}_{ml}, b_m, l = 1, \ldots, L, m = 1, \ldots, M$;

`Output` : an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ with $\sum_{l=1}^{L} \text{rank}^2(\boldsymbol{X}_l^\star) \leq M$;

1: solve the separable SDP (P0) finding $\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star$, with arbitrary ranks;

2: evaluate $R_l = \text{rank}(\boldsymbol{X}_l^\star)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L} R_l^2$;

3: `while` $U > M$ `do`

4: decompose $\boldsymbol{X}_l = \boldsymbol{V}_l \boldsymbol{V}_l^H$, $l = 1, \ldots, L$;

5: find a nonzero solution $(\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_L)$ of the system of linear equations:

$$\sum_{l=1}^{L} \boldsymbol{V}_l^H \boldsymbol{A}_{ml} \boldsymbol{V}_l \bullet \boldsymbol{\Delta}_l = 0, \quad m = 1, \ldots, M$$

where $\boldsymbol{\Delta}_l$ is a $R_l \times R_l$ Hermitian matrix for all l;

6: evaluate the eigenvalues $\delta_{l1}, \ldots, \delta_{lR_l}$ of $\boldsymbol{\Delta}_l$ for $l = 1, \ldots, L$;

7: determine l_0 and k_0 such that

$$|\delta_{l_0 k_0}| = \max\{|\delta_{lk}| : 1 \leq k \leq R_l, 1 \leq l \leq L\}.$$

8: compute $\boldsymbol{X}_l^\star = \boldsymbol{V}_l (\boldsymbol{I}_{R_l} - 1/\delta_{l_0 k_0} \boldsymbol{\Delta}_l) \boldsymbol{V}_l^H$, $l = 1, \ldots, L$;

9: evaluate $R_l = \text{rank}(\boldsymbol{X}_l^\star)$, $l = 1, \ldots, L$, and $U = \sum_{l=1}^{L} R_l^2$;

10: `end while`

It is noteworthy that since the feasibility and optimality are always satisfied in the each iterative step of the purification process, Algorithm 1 can be applied to problem (P0) with or without the I groups of individual semidefinite shaping constraints. Indeed, there is another way to more efficiently cope with the groups of individual semidefinite shaping constraints. The inequality constraint $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l \geq 0$ is redundant and can be removed if $\boldsymbol{D}_{il} \succeq \mathbf{0}$, and can only be satisfied with equality if $\boldsymbol{D}_{il} \preceq \mathbf{0}$. Observe that $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l = 0$ is equivalent $(-\boldsymbol{D}_{il}) \bullet \boldsymbol{X}_l = 0$. We thus preprocess the individual shaping constraints as follows: (i) Discard the constraints $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l \geq 0$ with $\boldsymbol{D}_{il} \succeq \mathbf{0}$ (i.e., by setting these $\boldsymbol{D}_{il} = \mathbf{0}$), and (ii) set $\boldsymbol{D}_{il} := -\boldsymbol{D}_{il}$ if $\boldsymbol{D}_{il} \preceq \mathbf{0}$, and (iii) group them into

$$\boldsymbol{D}_l \bullet \boldsymbol{X}_l = 0, \quad l = 1, \ldots, L \tag{22}$$

$$\text{(D0)} \begin{cases} \underset{y_m, z_{il}}{\text{maximize}} & \sum_{m=1}^{M} y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{M} y_m \boldsymbol{A}_{ml} - \sum_{i=1}^{I} z_{il} \boldsymbol{D}_{il} \succeq \mathbf{0}, \quad l = 1, \ldots, L, \\ & y_m \trianglerighteq_m^* 0, \quad m = 1, \ldots, M, \\ & z_{il} \geq 0, \ \forall l \notin \mathcal{E}_i, \quad i = 1, \ldots, I. \end{cases} \tag{15}$$

where $\boldsymbol{D}_l = \sum_{i=1}^{I} \boldsymbol{D}_{il} \succeq \boldsymbol{0}$. In words, after this preprocessing, the I groups of individual semidefinite shaping constraints are turned equivalently into one group of individual constraints. It is seen that if $\boldsymbol{X}_l = \boldsymbol{V}_l \boldsymbol{V}_l^H$, then the columns of $\boldsymbol{V}_l$ are in $\text{Null}(\boldsymbol{D}_l)$, that is, $\boldsymbol{V}_l$ can be expressed as $\boldsymbol{V}_l = \tilde{\boldsymbol{U}}_l \tilde{\boldsymbol{V}}_l$, where the orthonormal columns of $\tilde{\boldsymbol{U}}_l$ span $\text{Null}(\boldsymbol{D}_l)$. Hence, by replacing $\boldsymbol{X}_l = \tilde{\boldsymbol{U}}_l \tilde{\boldsymbol{V}}_l \tilde{\boldsymbol{V}}_l^H \tilde{\boldsymbol{U}}_l^H$ in the optimization problem and optimizing over $\tilde{\boldsymbol{V}}_l$, the dimension of the search space is decreased and the individual semidefinite shaping constraints are automatically satisfied.

Theorem 3.1 provides an upper bound of the rank profile of a solution which can be purified. Interestingly, it turns out that for some cases where the constraints of (P0) are not "too much," there is only one rank profile satisfying (21), for example, the case of a rank-one optimal solution when the number M of constraints is no more than $L+2$, as stated in the proposition here.

Proposition 3.2: Suppose that the parameters $\boldsymbol{D}_{il}$ comply with (17). Suppose that the primal problem (P0) and the dual problem (D0) are solvable. Suppose also that any optimal solution of problem (P0) has no zero matrix component. If $M \le L+2$, then (P0) has an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ with each $\boldsymbol{X}_l^\star$ of rank one.

In particular, sufficient conditions guaranteeing that any optimal solution has no zero matrix component are the following (which in fact guarantee that any feasible point has no zero matrix component):

$$L \le M, \tag{23}$$

$$-\boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \ \forall l \neq m, \ m, l = 1, \ldots, L \tag{24}$$

$$b_m > 0, \ m = 1, \ldots, L. \tag{25}$$

It thus follows from Proposition 3.2 that (P0) has a rank-one optimal solution if its parameters satisfy the conditions (23)–(25) and $M \le L+2$.

It is easily verified that the beamforming SDP relaxation problem (SDR) of Section II, with $M = L+2$ and $I \ge 0$, fulfills conditions (23)–(25), thus it has an optimal solution of rank one. In other words, the OBP is solvable with L SINR constraints, two additional soft-shaping interference constraints and I groups of individual semidefinite shaping constraints (an optimal solution is obtained by solving its SDP relaxation problem (SDR) and calling the rank reduction procedure described in Algorithm 1), provided that the SDP relaxation of (OBP) and its dual are solvable.[3]

[3]This means that there is no gap between the SDP relaxation and the original (OBP), and we see the relation between the solvability of the SDP relaxation and the solvability of the problem (OBP): If the SDP relaxation is solvable, then the original (OBP) is solvable, and vice versa (the solvability of the SDP relaxation follows from the SDP strong duality theorem since the dual of the SDP relaxation is strictly feasible and the SDP relaxation is feasible due to the feasibility of the original (OBP)).

B. An Application in Multicast Downlink Beamforming

We find another application of Proposition 3.2 in a scenario of multicast beamforming (e.g., see [10]). Consider a communication system with a single BS (transmitter) equipped with a K-element antenna array and L receivers, each with a single antenna. Let $\boldsymbol{R}_l$ be the channel correlation matrix between the transmitter and receiver $l \in \{1, \ldots, L\}$. Each receiver listens to a single multicast stream $m \in \{1, \ldots, G\}$, where $1 \le G \le L$ is the total number of multicast groups $\{\mathcal{G}_1, \ldots, \mathcal{G}_G\}$ with $\mathcal{G}_m$ being the index set of the receivers participating in multicast group m. $\{\mathcal{G}_1, \ldots, \mathcal{G}_G\}$ has the following properties: $\mathcal{G}_m \cap \mathcal{G}_n = \emptyset$, $m \neq n$, $\cup_m \mathcal{G}_m = \{1, \ldots, L\}$, and $\sum_m |\mathcal{G}_m| = L$. The transmitted signal at the BS is $\sum_{m=1}^{G} \boldsymbol{w}_m s_m(t)$, where $\boldsymbol{w}_m \in \mathbb{C}^K$ is the beamforming vector for group m and $s_m(t)$ is the data stream directed to receivers in group m (the data streams are assumed to be temporally white with zero mean and unit variance and mutually independent). The optimal design of multicast transmit beamforming is formulated into the minimization problem of the total transmission power at the BS subject to meeting prescribed SINR constraints ρ_l at each of the L receivers, as well as some soft-shaping interference constraints [cf. (6)]: Problem (26) shown at the bottom of the page, where $\boldsymbol{S}_{im} \succeq \boldsymbol{0}, \forall i, m$.

It is known from [9] that problem (26) is NP-hard when $G = 1$ and $I = 0$. When $G = L$, problem (26) coincides with a single-BS instance of problem (OBP). It follows from Proposition 3.2 that the following instances of multicast beamforming problem (26) are solvable with parameters: (I) $|\mathcal{G}_m| = 1$, $m = 1, \ldots, G-2$, $|\mathcal{G}_{G-1}| = |\mathcal{G}_G| = 2$, $\tau_i = 0, \forall i$; (II) $|\mathcal{G}_m| = 1$, $m = 1, \ldots, G-1$, $|\mathcal{G}_G| = 3$, $\tau_i = 0, \forall i$; (III) $|\mathcal{G}_m| = 1$, $m = 1, \ldots, G-1$, $|\mathcal{G}_G| = 2$, $\tau_1 > 0$, $\tau_i = 0, \forall i \ge 2$.

IV. Separable Semidefinite Programming From the Dual Perspective

The previous section focuses on rank-constrained solutions of a separable SDP from the primal perspective, in the sense that in each rank reduction step, an optimal solution of the primal separable SDP is updated such that the rank sum $\sum_{l=1}^{L} \text{rank}(\boldsymbol{X}_l^\star)$ is decreased at least by one, while the optimal solution of its dual remains the same. At the end of the iterative procedure, it outputs another solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ of rank satisfying $\sum_{l=1}^{L} \text{rank}^2(\boldsymbol{X}_l^\star) \le M$.

It turns out that the rank of each matrix component $\boldsymbol{X}_l^\star$ is however under no control, namely, the rank profile $(\text{rank}(\boldsymbol{X}_1^\star), \ldots, \text{rank}(\boldsymbol{X}_L^\star))$ is not known exactly *a priori*. This section aims at another efficient algorithm for a separable SDP from the dual perspective, in the sense that an optimal

$$\begin{cases} \underset{\boldsymbol{w}_m \in \mathbb{C}^K, m=1,\ldots,G}{\text{minimize}} & \sum_{m=1}^{G} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} & \frac{\boldsymbol{w}_m^H \boldsymbol{R}_l \boldsymbol{w}_m}{\sum_{n \neq m} \boldsymbol{w}_n^H \boldsymbol{R}_l \boldsymbol{w}_n + \sigma_l^2} \ge \rho_l, \ \forall l \in \mathcal{G}_m, \quad m = 1, \ldots, G, \\ & \sum_{m=1}^{G} \boldsymbol{w}_m^H \boldsymbol{S}_{im} \boldsymbol{w}_m \le \tau_i, \ i = 1, \ldots, I. \end{cases} \tag{26}$$

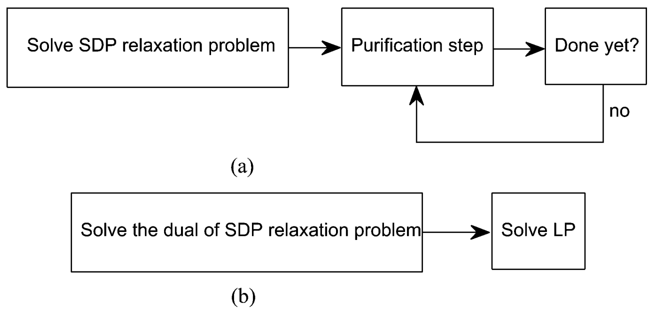

Fig. 1. The primal approach ffis based on iterative purification steps (rank reduction steps) while the dual approach is based on solving an LP in one single step. (a) Primal approach. (b) Dual approach.

solution of its dual problem is explored so as to produce a rank-constrained (e.g., rank-one) solution of the (primal) separable SDP. Interestingly, the rank profile of the output solution is controllable by the user in some sense as it will be later described, and the desired optimal solution can be found by solving an LP in one single step, which is in contrast to the iterative rank reduction steps (i.e., the purification steps) described in Algorithm 1 derived from the primal perspective (see Fig. 1 for a pictorial comparison).

Consider the following separable SDP and its dual:

$$\text{(P)}\begin{cases} \underset{\boldsymbol{X}_1,\dots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l \\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m, \quad m=1,\dots,M, \\ & \boldsymbol{X}_l \succeq \boldsymbol{0}, \quad l=1,\dots,L \end{cases} \tag{27}$$

and

$$\text{(D)}\begin{cases} \underset{y_1,\dots,y_M}{\text{maximize}} & \sum_{m=1}^{M} y_m b_m \\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{M} y_m \boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \quad l=1,\dots,L, \\ & y_m \trianglerighteq_m^* 0, \quad m=1,\dots,M \end{cases} \tag{28}$$

where

$$\trianglerighteq_m \in \begin{cases} \{\geq, =\}, & m=1,\dots,L, \\ \{\leq, =\}, & m=L+1,\dots,M \end{cases} \tag{29}$$

and $\trianglerighteq_m^*$, $m=1,\dots,M$, are defined in (16). The complementary conditions for the primal and dual SDPs (P) and (D) are

$$\boldsymbol{Z}_l \bullet \boldsymbol{X}_l = 0, \; l=1,\dots,L \tag{30}$$

$$y_m \left(\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l - b_m \right) = 0, \; m=1,\dots,M \tag{31}$$

where $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_L)$ and $(y_1,\dots,y_M,\boldsymbol{Z}_1,\dots,\boldsymbol{Z}_L)$ are feasible points of (P) and (D), respectively. In other words, if a feasible primal-dual pair $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_L; y_1,\dots,y_M, \boldsymbol{Z}_1,\dots,\boldsymbol{Z}_L)$ satisfies (30) and (31), then the pair is optimal.

A. *Properties of the Separable SDPs (P) and (D)*

Assume that the parameters of problem (P) satisfy

$$-\boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \quad \forall m \neq l, \; m, \; l=1,\dots,L \tag{32}$$

$$b_m > 0, \quad m=1,\dots,L. \tag{33}$$

In particular, the SINR constraints of OBP [cf. (12)] have their parameters fulfilling (32) and (33). Due to the assumptions (32) and (33), problems (P) and (D) have some specific properties, the proofs of which are based on feasibility and complementarity of (P) and (D).

Proposition 4.1: Suppose that the parameters of (P) satisfy (32) and (33), and that both SDPs (P) and (D) are solvable, with solutions $(\boldsymbol{X}_1^\star,\dots,\boldsymbol{X}_L^\star)$ and $(y_1^\star,\dots,y_M^\star,\boldsymbol{Z}_1^\star,\dots,\boldsymbol{Z}_L^\star)$, respectively. Then

(i) $\boldsymbol{X}_l^\star \neq \boldsymbol{0}$, $l=1,\dots,L$;

(ii) $\boldsymbol{Z}_l^\star \succeq \boldsymbol{0}$, and $\not\succ \boldsymbol{0}$, $l=1,\dots,L$.

Proof: See Appendix B. ■

Let us further assume that the parameters satisfy

$$\boldsymbol{C}_l \succ \boldsymbol{0}, \quad l=1,\dots,L \tag{34}$$

and

$$\boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \quad \trianglerighteq_m \in \{\leq\}, \; m=L+1,\dots,M, \; l=1,\dots,L. \tag{35}$$

Note that (35) implies that $b_m \geq 0$, $m=L+1,\dots,M$ (assuming problem (P) is feasible). Particularly, the soft-shaping interference constraints of (OBP) comply with assumption (35), and the parameters of the objective function of (OBP) satisfy the assumption (34).

Proposition 4.2: Suppose that the parameters of (P) satisfy (32)–(35), and that both SDPs (P) and (D) are solvable, with solutions $(\boldsymbol{X}_1^\star,\dots,\boldsymbol{X}_L^\star)$ and $(y_1^\star,\dots,y_M^\star,\boldsymbol{Z}_1^\star,\dots,\boldsymbol{Z}_L^\star)$, respectively. Then, $y_m^\star > 0$, $\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star = b_m$, $m=1,\dots,L$.

Proof: See Appendix C. ■

Now, let us investigate some properties of the optimal solution set of dual SDP (D). We define the set shown in (36) at the bottom of the page, given $(y_{L+1},\dots,y_M) \in \mathbb{R}^{M-L}$. In the particular case of $M=L$, the set defined in (36) reduces to (37), shown at the bottom of the page, and it is seen that $\mathcal{Y} = \mathcal{Y}(0,\dots,0)$ for $M > L$.

Assume that $(y_1^\star,\dots,y_M^\star)$ is an optimal solution of (D) and (P) is solvable and the parameter assumptions (32) and (33) are satisfied, it thus follows from Proposition 4.1 that

$$\mathcal{Y}(y_{L+1},\dots,y_M) = \left\{ (y_1,\dots,y_L) \in \mathbb{R}_+^L \,|\, \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{M} y_m \boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \text{ and } \not\succ \boldsymbol{0}, \; l=1,\dots,L \right\} \tag{36}$$

$$\mathcal{Y} = \left\{ (y_1,\dots,y_L) \in \mathbb{R}_+^L \,|\, \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{L} y_m \boldsymbol{A}_{ml} \succeq \boldsymbol{0}, \text{ and } \not\succ \boldsymbol{0}, \; l=1,\dots,L \right\} \tag{37}$$

$(y_1^\star,\ldots,y_L^\star) \in \mathcal{Y}(y_{L+1}^\star,\ldots,y_M^\star)$. Additionally assume that the parameter assumptions (34) and (35) are fulfilled, it then follows from the complementary condition (30) that the optimal solutions of (P) must be of *rank one* if the matrices $\boldsymbol{Z}_l$ in $\mathcal{Y}(y_{L+1}^\star,\ldots,y_M^\star)$ are of rank $K-1$ (it is the case, for instance, when matrices $\boldsymbol{A}_{ll}$ are of rank one, i.e., $\boldsymbol{A}_{ll} = \boldsymbol{h}_l\boldsymbol{h}_l^H$, where $\boldsymbol{h}_l$, $l=1,\ldots,L$, are the channel vectors).

In addition, the following result gives an explicit characterization of the elements $(y_1,\ldots,y_L)$ of set $\mathcal{Y}(y_{L+1},\ldots,y_M)$, where $(y_{L+1},\ldots,y_M) \in -\mathbb{R}_+^{M-L}$ is given.

Proposition 4.3: Suppose that $(y_{L+1},\ldots,y_M) \in -\mathbb{R}_+^{M-L}$ is given and $(y_1,\ldots,y_L) \in \mathcal{Y}(y_{L+1},\ldots,y_M)$, where the parameters $\{\boldsymbol{A}_{ml}\}$, $\{\boldsymbol{C}_l\}$ fulfill (32), (34) and (35). Then,

(i) $y_l \geq 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2}\boldsymbol{A}_{ll}\boldsymbol{E}_l^{-1/2}) > 0$, $l=1,\ldots,L$, where $\boldsymbol{E}_l = \boldsymbol{C}_l - \sum_{m\neq l,\, m=1}^M y_m\boldsymbol{A}_{ml}$;

(ii) $y_l = 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2}\boldsymbol{A}_{ll}\boldsymbol{E}_l^{-1/2})$, $l=1,\ldots,L$, amounts to $\boldsymbol{Z}_l \succeq \boldsymbol{0}$ and $\not\succ \boldsymbol{0}$, $l=1,\ldots,L$;

(iii) the set $\mathcal{Y}(y_{L+1},\ldots,y_M)$ contains only one point with $y_l = 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2}\boldsymbol{A}_{ll}\boldsymbol{E}_l^{-1/2})$, $l=1,\ldots,L$.

Proof: See Appendix D. ■

We remark that when $M=L$, it follows from Proposition 4.3 (iii) that the optimal solution of problem (D) is unique. In particular, the dual of the OBP with only SINR constraints (no soft-shaping constraints) has an unique optimal solution.

B. Rank-One Solution of Separable SDP Via Linear Programming

Here, we consider the separable SDP (P) with the first L constraints only (i.e., $M=L$) and propose an efficient algorithm by solving a linear program based on observations of the previous subsection. From Theorem 3.1, we know that for $M=L$ there always exists a rank-one solution, which we are about to present a new way to output. Other interesting results on problem (P) involving more types of constraints will be presented in the next section. The problem and its dual are

$$(\text{P1})\begin{cases}\underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^L \boldsymbol{C}_l \bullet \boldsymbol{X}_l\\ \text{subject to} & \sum_{l=1}^L \boldsymbol{A}_{ml}\bullet\boldsymbol{X}_l \trianglerighteq_m b_m, \quad m=1,\ldots,L,\\ & \boldsymbol{X}_l\succeq\boldsymbol{0},\ l=1,\ldots,L\end{cases} \tag{38}$$

and

$$(\text{D1})\begin{cases}\underset{y_1,\ldots,y_L}{\text{maximize}} & \sum_{m=1}^L y_m b_m\\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^L y_m\boldsymbol{A}_{ml}\succeq\boldsymbol{0}, \quad l=1,\ldots,L,\\ & y_m \trianglerighteq_m^* 0, \quad m=1,\ldots,L\end{cases} \tag{39}$$

where $\trianglerighteq_m$ and $\trianglerighteq_m^*$ are defined in (29) and (16), respectively, and we assume that the parameters of problem (P1) satisfy conditions (32)–(34). Observe that the SDP relaxation problem of the OBP of Section II with only SINR constraints belongs to the class of problem (P1).

Suppose that both (P1) and (D1) are solvable, and let $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ and $(y_1^\star,\ldots,y_L^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ be optimal solutions of (P1) and (D1), respectively. It follows from Proposition 4.1 that $\boldsymbol{X}_l^\star \neq \boldsymbol{0}$, $\boldsymbol{Z}_l^\star \succeq \boldsymbol{0}$, and $\not\succ \boldsymbol{0}$, $l=1,\ldots,L$, and from Proposition 4.2 that $y_m^\star > 0$, $\sum_{l=1}^L \boldsymbol{A}_{ml}\bullet\boldsymbol{X}_l^\star = b_m$, $m=1,\ldots,L$. In other words, we can safely change the general inequalities $\trianglerighteq_m b_m$ of problem (P1) into inequalities $\geq b_m$ without loss of generality. Therefore, from now on we will consider each $\trianglerighteq_m \in \{\geq\}$ and the corresponding $\trianglerighteq_m^* \in \{\geq\}$ in (38) and (39).

It is clear that the set $\mathcal{Y}$ defined in (37) is equivalent to (40), shown at the bottom of the page. Given the solution $(y_1^\star,\ldots,y_L^\star)$, the corresponding $\boldsymbol{u}_l$ in (40) can be taken as follows: $\boldsymbol{u}_l \in \text{Null}(\boldsymbol{Z}_l^\star)$, $\|\boldsymbol{u}_l\|=1$, for $l=1,\ldots,L$. In an alternative way, one may take $\boldsymbol{u}_l \in \text{Range}(\boldsymbol{X}_l^\star)$ and $\|\boldsymbol{u}_l\|=1$. Indeed, since $\boldsymbol{X}_l^\star \neq \boldsymbol{0}$ and $\boldsymbol{Z}_l^\star\boldsymbol{X}_l^\star = \boldsymbol{0}$ (from the complementary condition (30)), it follows that $\text{Range}(\boldsymbol{X}_l^\star) \subseteq \text{Null}(\boldsymbol{Z}_l^\star)$ and $\text{Range}(\boldsymbol{X}_l^\star) \neq \emptyset$.

Now, let us investigate how to retrieve a rank-one solution of (P1) from $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$, each component of which could be of arbitrary rank. We formulate the following LP shown in (41) at the bottom of the page. Note that vectors $\boldsymbol{u}_l$ are fixed parameters and not part of the optimization. This LP possesses the key properties that it is feasible (since the solution $(y_1^\star,\ldots,y_L^\star)$ of (D1) is feasible to (LP1)) and bounded below (since $y_m \geq 0$ and $b_m > 0, \forall m$). Thus, it follows by the duality theorem of LP

$$\left\{(y_1,\ldots,y_L)\in\mathbb{R}_+^L \middle| \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^L y_m\boldsymbol{A}_{ml}\succeq\boldsymbol{0}, \exists\boldsymbol{u}_l,\ \text{s.t. } \boldsymbol{u}_l^H\boldsymbol{Z}_l\boldsymbol{u}_l \leq 0,\ \|\boldsymbol{u}_l\|=1,\ l=1,\ldots,L\right\}. \tag{40}$$

$$(\text{LP1})\begin{cases}\underset{y_1,\ldots,y_L}{\text{minimize}} & \sum_{m=1}^L y_m b_m\\ \text{subject to} & \boldsymbol{u}_l^H\boldsymbol{C}_l\boldsymbol{u}_l - \sum_{m=1}^L y_m\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l \leq 0, \quad l=1,\ldots,L,\\ & y_m\geq 0, \quad m=1,\ldots,L.\end{cases} \tag{41}$$

(for example, see [16, Theorem 1.2.2]) that both (LP1) and its dual (DLP1) are solvable

$$(\text{DLP1})\begin{cases}\underset{x_1,\ldots,x_L}{\text{maximize}} & \sum_{l=1}^{L}\boldsymbol{u}_l^H\boldsymbol{C}_l\boldsymbol{u}_l x_l\\ \text{subject to} & \sum_{l=1}^{L}\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l x_l\le b_m,\ m=1,\ldots,L\\ & x_l\ge 0,\quad l=1,\ldots,L.\end{cases}\tag{42}$$

Let $(y_1^\star,\ldots,y_L^\star)$ and $(x_1^\star,\ldots,x_L^\star)$ be optimal solutions of (LP1) and (DLP1), respectively. Then, the complementary conditions of (LP1) and (DLP1) are satisfied

$$x_l^\star\left(\boldsymbol{u}_l^H\boldsymbol{C}_l\boldsymbol{u}_l-\sum_{m=1}^{L}y_m^\star\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l\right)=0,\quad l=1,\ldots,L\tag{43}$$

$$y_m^\star\left(\sum_{l=1}^{L}\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l x_l^\star-b_m\right)=0,\quad m=1,\ldots,L.\tag{44}$$

These two LPs are very useful and we will see that problem (DLP1) provides rank-one optimal solutions to (P1). To proceed, let us denote

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{u}_1^H\boldsymbol{A}_{11}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_L^H\boldsymbol{A}_{1L}\boldsymbol{u}_L\\ \vdots & \ddots & \vdots\\ \boldsymbol{u}_1^H\boldsymbol{A}_{L1}\boldsymbol{u}_1 & \cdots & \boldsymbol{u}_L^H\boldsymbol{A}_{LL}\boldsymbol{u}_L\end{bmatrix}\text{ and }\boldsymbol{b}=\begin{bmatrix}b_1\\ \vdots\\ b_L\end{bmatrix}.\tag{45}$$

Theorem 4.4: Suppose that the parameters of (P1) satisfy (32)–(34). Suppose that both (P1) and (D1) are solvable and let $(y_1^\star,\ldots,y_L^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ be an optimal solution of (D1). Take any vectors $\boldsymbol{u}_l\in\text{Null}(\boldsymbol{Z}_l^\star)$ with $\|\boldsymbol{u}_l\|=1$, $\forall l$, and formulate the two LPs (LP1) and (DLP1). Let $(x_1^\star,\ldots,x_L^\star)$ be an optimal solution of (DLP1). Then,

(i) $\sum_{l=1}^{L}\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l x_l^\star=b_m$, $m=1,\ldots,L$;
(ii) $x_l^\star>0$, $l=1,\ldots,L$;
(iii) $\boldsymbol{A}$ defined in (45) is invertible, and $\boldsymbol{x}^\star=\boldsymbol{A}^{-1}\boldsymbol{b}$;
(iv) $v(\text{D1})=v(\text{LP1})=v(\text{DLP1})=v(\text{P1})$, where $v(\cdot)$ represents the optimal value of problem $(\cdot)$; and $(x_1^\star\boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star\boldsymbol{u}_L\boldsymbol{u}_L^H)$ is optimal for (P1).

Proof: See Appendix E. ■

It follows from Theorem 4.4 that a rank-one solution of (P1) can be obtained by solving (DLP1), the formulation of which is based on an optimal solution of (D1), and that (DLP1) has always the closed-form solution. In other words, a rank-one solution of SDP (P1) can be found by solving its dual and an LP. Algorithm 2 summarizes the procedure to generate a rank-one solution of (P1).

Algorithm 2: Procedure for Rank-One Solution of Separable SDP

`Input` : $\boldsymbol{C}_l,\boldsymbol{A}_{ml},b_m,l=1,\ldots,L,m=1,\ldots,L$, satisfying **(32)–(34)**;

`Output` : $\boldsymbol{X}_l^\star$ with $\text{rank}(\boldsymbol{X}_l^\star)=1,l=1,\ldots,L$;

1:solve the dual SDP (D1), finding $(y_1^\star,\ldots,y_L^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$;

2:take unit-norm vectors $\boldsymbol{u}_l\in\text{Null}(\boldsymbol{Z}_l^\star),l=1,\ldots,L$;

3:form the matrix $\boldsymbol{A}$ as in **(45)**, and compute $\boldsymbol{x}^\star=\boldsymbol{A}^{-1}\boldsymbol{b}$;

4:output $\boldsymbol{X}_l^\star=x_l^\star\boldsymbol{u}_l\boldsymbol{u}_l^H,l=1,\ldots,L$.

We remark that when solving (P1) with the parameter assumptions (32) and (33), Algorithm 1 (from the primal perspective) gives a solution of rank $\sum_{l=1}^{L}\text{rank}^2(\boldsymbol{X}_l^\star)\le L$, i.e., a rank-one solution (since $\boldsymbol{X}_l^\star\ne\boldsymbol{0}$, $\forall l$), while the additional parameter assumption (34) allows us to find a rank-one solution of (P1) by Algorithm 2 (from the dual perspective). As shall be seen in the next subsection, the dual approach can also provide a solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ of (P1) with a more arbitrary rank profile satisfying $1\le\text{rank}(\boldsymbol{X}_l^\star)\le P_l$, $\forall l$, where $(P_1,\ldots,P_L)$ is a prefixed rank profile.

C. Separable SDP's Solution With a Given Rank Profile

In this subsection, we show that the above LP approach can also be used to provide an optimal solution to the separable SDP (P1) satisfying a prefixed rank profile, not merely rank-one solution. One motivation of imposing a rank profile lies in the robust design of the multiple transmit beamforming architecture for a MIMO communication with transmission using OSTBC, as described in Section II-A. Besides, another motivation comes from the setting of a cognitive MIMO radio network; in order to transmit over the same frequency band but without interfering, a secondary transmitter has an antenna array and uses multiple beamforming to put nulls over the directions identifying the primary receivers and make the degradation induced on the primary users performance null or tolerable (see [14]).

Suppose that $(y_1^\star,\ldots,y_L^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ is an optimal solution of (D1). Let $D_l=\dim(\text{Null}(\boldsymbol{Z}_l^\star))$, $l=1,\ldots,L$. It follows from Proposition 4.1 that $\boldsymbol{Z}_l^\star\succeq\boldsymbol{0}$ and $\not\succ\boldsymbol{0}$, thus $D_l\ge 1$ for $l=1,\ldots,L$. Suppose that a rank profile $(P_1,\ldots,P_L)$ is given. The goal now is to find an optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ of problem (P1) such that $1\le\text{rank}(\boldsymbol{X}_l^\star)\le P_l$, $l=1,\ldots,L$. Since any optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ of (P1), together with $(\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$, fulfills the complementary condition (30), i.e., $\boldsymbol{Z}_l^\star\boldsymbol{X}_l^\star=\boldsymbol{0}$, $\forall l$, the rank of $\boldsymbol{X}_l^\star$ (i.e., the dimension of the range of $\boldsymbol{X}_l^\star$) cannot be more than D_l. Based on the observation, we substitute P_l with $\min\{P_l,D_l\}$ for each l. In particular, when the rank profile is specified by $P_l=1$, $\forall l$, we can output a desired solution of (P1) with Algorithm 2.

Take any unit norm vectors $\boldsymbol{u}_{lk}\in\text{Null}(\boldsymbol{Z}_l^\star)$, $l=1,\ldots,L$, $k=1,\ldots,P_l$, such that $\{\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lP_l}\}$ are orthogonal vectors, for each l, and formulate the following LP shown in (46) at the bottom of the next page. Note that a solution $(y_1^\star,\ldots,y_M^\star)$ of (D1) is feasible for (LP2), and (LP2) is bounded below (since $b_m>0$ and $y_m\ge 0$, $\forall m$), and it follows again by the LP strong duality theorem that (LP2) and its dual problem (DLP2), shown in (47) at the bottom of the next page, are solvable.

Similarly to Theorem 4.4, an optimal solution of (DLP2) yields a rank-constrained solution to (P1).

Theorem 4.5: Suppose that the parameters of (P1) satisfy (32)–(34), and that both (P1) and (D1) are solvable. Let P_l, $l=1,\ldots,L$, be given positive integers, and let $(y_1^\star,\ldots,y_L^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ be an optimal solution of (D1). Take any P_l unit-norm vectors $\boldsymbol{u}_{lk}\in\text{Null}(\boldsymbol{Z}_l^\star)$ such that $\{\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lP_l}\}$ are orthogonal, $\forall l$, and formulate the LPs (LP2) and (DLP2). Let $(x_{11}^\star,\ldots,x_{1P_1}^\star,\ldots,x_{L1}^\star,\ldots,x_{LP_L}^\star)$ be an optimal solution of (DLP2). Then,

(i) $\sum_{l=1}^{L}\sum_{k=1}^{P_l}\boldsymbol{u}_{lk}^H\boldsymbol{A}_{ml}\boldsymbol{u}_{lk}x_{lk}^\star=b_m$, $m=1,\ldots,L$;

(ii) at least one of $\{x^\star_{l1}, \ldots, x^\star_{lP_l}\}$ is positive, for any $l \in \{1, \ldots, L\}$;

(iii) $v(\text{D1}) = v(\text{LP2}) = v(\text{DLP2}) = v(\text{P1})$, and $\left(\sum_{k=1}^{P_1} x^\star_{1k}\boldsymbol{u}_{1k}\boldsymbol{u}_{1k}^H, \ldots, \sum_{k=1}^{P_L} x^\star_{Lk}\boldsymbol{u}_{Lk}\boldsymbol{u}_{Lk}^H\right)$ is an optimal solution of (P1).

Proof: See Appendix F. ■

It is seen easily that the optimal solution obtained in Theorem 4.5 has rank no more than P_l and at least one, $\forall l$. Algorithm 3 summarizes the procedure to produce an optimal solution of (P1) complying with a prescribed rank profile. Observe that the specified rank profile $(P_1, \ldots, P_L)$ can be achieved only if $P_l \le D_l\ \forall l$; otherwise, we can only achieve $\min\{P_l, D_l\}$ as indicated in Algorithm 3.

Algorithm 3: Procedure for Solution of Separable SDP Problem With a Given Rank Profile

`Input` : $\boldsymbol{C}_l, \boldsymbol{A}_{ml}, b_m, m, l = 1, \ldots, L$, satisfying **(32)–(34)**; the rank profile $(P_1, \ldots, P_L)$;

`Output` : $\boldsymbol{X}^\star_l$ with $1 \le \operatorname{rank}(\boldsymbol{X}^\star_l) \le P_l, l = 1, \ldots, L$;

1:solve the dual SDP (D1), finding $(y^\star_1, \ldots, y^\star_L, \boldsymbol{Z}^\star_1, \ldots, \boldsymbol{Z}^\star_L)$;

2:$D_l := \dim(\text{Null}(\boldsymbol{Z}^\star_l)), l = 1, \ldots, L$;

3:substitute P_l with $\min\{P_l, D_l\}$;

4:take any vectors $\boldsymbol{u}_{lk} \in \text{Null}(\boldsymbol{Z}^\star_l), k = 1, \ldots, P_l$, such that $\{\boldsymbol{u}_{l1}, \ldots, \boldsymbol{u}_{lP_l}\}$ are orthonormal, for $l = 1, \ldots, L$;

5:form the linear program (DLP2), and solve it, finding $x^\star_{lk}$, $l = 1, \ldots, L, k = 1, \ldots, P_l$;

6:output $\boldsymbol{X}^\star_l = \sum_{k=1}^{P_l} x^\star_{lk}\boldsymbol{u}_{lk}\boldsymbol{u}_{lk}^H, l = 1, \ldots, L$.

V. Separable SDP With Individual Shaping Constraints Via Linear Programming

We consider now a separable SDP problem with additional individual shaping constraints:

$$(\text{P2})\begin{cases}\underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{minimize}} & \sum_{l=1}^{L} \boldsymbol{C}_l \bullet \boldsymbol{X}_l\\ \text{subject to} & \sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l \trianglerighteq_m b_m,\ m = 1, \ldots, L\\ & \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l = (\ge) 0, \forall l \in (\notin)\mathcal{E}_i, i = 1, \ldots, I\\ & \boldsymbol{X}_l \succeq \boldsymbol{0}, l = 1, \ldots, L\end{cases} \tag{48}$$

where the parameters $\boldsymbol{C}_l$, $\boldsymbol{A}_{ml}$, b_m, $\forall l, m$, comply with assumptions (32)–(34) and $\trianglerighteq_m \in \{=, \ge\}, \forall m$. Its dual problem is shown in (49) at the bottom of the page, where $\trianglerighteq^*_m$, $m = 1, \ldots, L$, are defined in (16). In this section, again resorting to an LP approach, we build an efficient algorithm to find a rank-one optimal solution of (P2), which is in contrast with the iterative rank reduction procedure for (P2) with two groups, $I = 2$, of individual shaping constraints as in [5].

Suppose that both problems (P2) and (D2) are solvable, and let $(\boldsymbol{X}^\star_1, \ldots, \boldsymbol{X}^\star_L)$, $(y^\star_1, \ldots, y^\star_L, z^\star_{11}, \ldots, z^\star_{IL}, \boldsymbol{Z}^\star_1, \ldots, \boldsymbol{Z}^\star_L)$ be optimal solutions of (P2) and (D2), respectively. It follows by the strong duality theorem (e.g., [16, Th. 1.7.1]) that they satisfy the complementary conditions (18)–(20) with $M = L$.

Let us highlight some properties of the primal and dual optimal solutions as next proposition.

Proposition 5.1: Suppose that the parameters of (P2) satisfy (32)–(34), and that both (P2) and (D2) are

$$(\text{LP2})\begin{cases}\underset{y_1,\ldots,y_L}{\text{minimize}} & \sum_{m=1}^{L} y_m b_m\\ \text{subject to} & \boldsymbol{u}_{lk}^H\boldsymbol{C}_l\boldsymbol{u}_{lk} - \sum_{m=1}^{L} y_m \boldsymbol{u}_{lk}^H\boldsymbol{A}_{ml}\boldsymbol{u}_{lk} \le 0, \quad l = 1, \ldots, L, \quad k = 1, \ldots, P_l,\\ & y_m \ge 0, \quad m = 1, \ldots, L.\end{cases} \tag{46}$$

$$(\text{DLP2})\begin{cases}\underset{\{x_{lk}\}}{\text{maximize}} & \sum_{l=1}^{L}\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H\boldsymbol{C}_l\boldsymbol{u}_{lk}x_{lk}\\ \text{subject to} & \sum_{l=1}^{L}\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H\boldsymbol{A}_{ml}\boldsymbol{u}_{lk}x_{lk} \le b_m, \quad m = 1, \ldots, L,\\ & x_{lk} \ge 0, \quad l = 1, \ldots, L, \quad k = 1, \ldots, P_l.\end{cases} \tag{47}$$

$$(\text{D2})\begin{cases}\underset{y_m,\, z_{il}}{\text{maximize}} & \sum_{m=1}^{L} y_m b_m\\ \text{subject to} & \boldsymbol{Z}_l = \boldsymbol{C}_l - \sum_{m=1}^{L} y_m\boldsymbol{A}_{ml} - \sum_{i=1}^{I} z_{il}\boldsymbol{D}_{il} \succeq \boldsymbol{0}, \quad l = 1, \ldots, L,\\ & y_m \trianglerighteq^*_m 0, \quad m = 1, \ldots, L,\\ & z_{il} \ge 0, \forall l \notin \mathcal{E}_i, \quad i = 1, \ldots, I.\end{cases} \tag{49}$$

solvable, with solutions $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ and $(y_1^\star, \ldots, y_L^\star, z_{11}^\star, \ldots, z_{IL}^\star, \boldsymbol{Z}_1^\star, \ldots, \boldsymbol{Z}_L^\star)$, respectively. Then,

(i) $\boldsymbol{X}_l^\star \neq \boldsymbol{0}, l = 1, \ldots, L$;

(ii) $\boldsymbol{Z}_l^\star \succeq \boldsymbol{0}$, and $\not\succ \boldsymbol{0}$, $l = 1, \ldots, L$;

(iii) $y_m^\star > 0$ and $\sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star = b_m, m = 1, \ldots, L$.

Proof: See Appendix G. ■

From the above proposition, we observe that either changing all general inequalities $\trianglerighteq_m b_m$ to $\geq b_m$ or changing all $\trianglerighteq_m b_m$ to $= b_m$ will not lose any generality in problem (P2). Thus, from now on we consider every $\trianglerighteq_m \in \{\geq\}$ and the corresponding $\trianglerighteq_m^* \in \{\geq\}$ in (48) and (49).

A. Individual Semidefinite Shaping Constraints

We consider separable SDP (P2) with individual shaping constraints where each $\boldsymbol{D}_{il}$ is semidefinite, i.e., all $\boldsymbol{D}_{il}$ satisfy (17).

Let $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ and $(y_1^\star, \ldots, y_L^\star, z_{11}^\star, \ldots, z_{IL}^\star, \boldsymbol{Z}_1^\star, \ldots, \boldsymbol{Z}_L^\star)$ be optimal solutions of (P2) and (D2), respectively. We can take a unit-norm vector $\boldsymbol{u}_l \in \text{Range}(\boldsymbol{X}_l^\star), \forall l$, since $\emptyset \neq \text{Range}(\boldsymbol{X}_l^\star) \subseteq \text{Null}(\boldsymbol{Z}_l^\star), \forall l$, and the formulate the LP problem shown in (50) at the bottom of the page, and its dual

$$\text{(DLP1)} \begin{cases} \underset{x_1,\ldots,x_L}{\text{maximize}} & \sum_{l=1}^L \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l x_l \\ \text{subject to} & \sum_{l=1}^L \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l x_l \leq b_m, \; m = 1, \ldots, L \\ & x_l \geq 0, \quad l = 1, \ldots, L. \end{cases} \tag{51}$$

We state some important properties of the LPs in the theorem.

Theorem 5.2: Suppose that the parameters of (P2) comply with (32), (33), and (17). Suppose that both (P2) and (D2) are solvable and let $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ and $(y_1^\star, \ldots, y_L^\star, \; z_{11}^\star, \ldots, z_{IL}^\star, \boldsymbol{Z}_1^\star, \ldots, \boldsymbol{Z}_L^\star)$ be optimal solutions of (P2) and (D2), respectively. Take any unit-norm vectors $\boldsymbol{u}_l \in \text{Range}(\boldsymbol{X}_l^\star), l = 1, \ldots, L$, and formulate the LPs (LP1) and (DLP1). Then,

(i) (LP1) is bounded below, and $(y_1^\star, \ldots, y_L^\star)$ is feasible for (LP1) (i.e., both (LP1) and (DLP1) are solvable).

Let $(x_1^\star, \ldots, x_L^\star)$ be an optimal solution of (DLP1) and suppose that $\boldsymbol{C}_l, l = 1, \ldots, L$, fulfill (34). Then,

(i) $\sum_{l=1}^L \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l x_l^\star = b_m, m = 1, \ldots, L$, and $x_l^\star > 0$, $l = 1, \ldots, L$, and $\boldsymbol{x}^\star = \boldsymbol{A}^{-1}\boldsymbol{b}$ where $\boldsymbol{A}$ is defined in (45);

(ii) $v(\text{D2}) = v(\text{LP1}) = v(\text{DLP1}) = v(\text{P2})$, and $(x_1^\star \boldsymbol{u}_1 \boldsymbol{u}_1^H, \ldots, x_L^\star \boldsymbol{u}_L \boldsymbol{u}_L^H)$ is an optimal solution of (P2).

Proof: See Appendix H. ■

Algorithm 4 summarizes the procedure to produce a rank-one optimal solution of (P2).

Algorithm 4: Procedure for Rank-One Solution of Separable SDP Problem With Individual Semidefinite Shaping Constraints

`Input` : $\boldsymbol{C}_l$, $\boldsymbol{A}_{ml}$, b_m, $\boldsymbol{D}_{il}$, $i = 1, \ldots, I, l = 1, \ldots, L$, $m = 1, \ldots, L$, satisfying **(32)–(34)** and **(17)**;

`Output` : an optimal solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$ of (P2), with $\text{rank}(\boldsymbol{X}_l^\star) = 1, \forall l$;

1:solve SDPs (P2) and (D2), finding solutions $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_L^\star)$, and $(y_1^\star, \ldots, y_M^\star, z_{11}^\star, \ldots, z_{IL}^\star, \boldsymbol{Z}_1^\star, \ldots, \boldsymbol{Z}_L^\star)$;

2:take unit-norm vectors $\boldsymbol{u}_l \in \text{Range}(\boldsymbol{X}_l^\star) \subseteq \text{Null}(\boldsymbol{Z}_l^\star)$, $l = 1, \ldots, L$;

3:form the matrix $\boldsymbol{A}$ as in **(45)**, and compute $\boldsymbol{x}^\star = \boldsymbol{A}^{-1}\boldsymbol{b}$;

4:output $\boldsymbol{X}_l^\star = x_l^\star \boldsymbol{u}_l \boldsymbol{u}_l^H, l = 1, \ldots, L$.

It is interesting to remark that the optimal value of problem (50) is no less than that of problem (41), due to the fact that $v(\text{P2}) \geq v(\text{P1})$ and the choice of $\boldsymbol{u}_l$ in (50) and (41) is different. Also, it is noted that a rank-one solution of (P1) with individual semidefinite shaping constraints can be output by applying the preprocess procedure introduced in Section III-A (the paragraph containing (22)) and Algorithm 2.

We point out that like in Section IV-C, it is possible to consider an arbitrary rank profile $(P_1, \ldots, P_L)$ with the difference that one has to solve (D2) and (P2) in Step 1 of Algorithm 3.

B. Individual Indefinite Shaping Constraints

In this subsection, we consider the separable SDP (P2) where some of the individual shaping constraints are indefinite. In particular, assume

$$\boldsymbol{D}_{il} \succeq \boldsymbol{0}, \text{ or } \boldsymbol{D}_{il} \preceq \boldsymbol{0}, \; i \geq 3, \; \forall l. \tag{52}$$

but with indefinite $\boldsymbol{D}_{il}, i = 1, 2, \forall l$, i.e., they could be any Hermitian matrix. We highlight that the SDP relaxation problem of the OBP of Section II with L SINR constraints, $I - 2$ null-shaping interference constraints and two groups of individual indefinite shaping constraints, belongs to this class of problem. By some specific rank-one matrix decomposition [17], we will show that problem (P2) with individual shaping constraints (52) have a rank-one solution, which can be generated from an optimal solution of (DLP1). Let us quote the useful matrix decomposition theorem in order to proceed.

Lemma 5.3 [17]: Suppose that $\boldsymbol{X}$ is a $K \times K$ complex Hermitian positive semidefinite matrix of rank R, and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are two $K \times K$ given Hermitian matrices. Then, there is a rank-one

$$\text{(LP1)} \begin{cases} \underset{y_1,\ldots,y_L}{\text{minimize}} & \sum_{m=1}^L y_m b_m \\ \text{subject to} & \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1}^L y_m \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l \leq 0, \quad l = 1, \ldots, L, \\ & y_m \geq 0, \quad m = 1, \ldots, L. \end{cases} \tag{50}$$

TABLE I
SOLVABLE INSTANCES OF OPTIMAL MULTIUSER DOWNLINK BEAMFORMING PROBLEM (OBP)

parameters	constraint types	references
$M=L,\ I=0$	L SINR constraints	Bengtsson-Ottersten [3]
$M=L,\ I=1$, $\boldsymbol{D}_{1l}$ indefinite	L SINR constraints, indefinite one group of individual indefinite shaping constraints	Hammarwall-Bengtsson-Ottersten [4]
$L\le M\le L+2$, $0\le I\le 2$ $\boldsymbol{D}_{il}$ semidefinite	L SINR constraints, up to two soft-shaping constraints up to two groups of individual semidefinite shaping constraints	Huang-Palomar [5]
$L\le M\le L+2$, $\boldsymbol{D}_{il}$ semidefinite	L SINR constraints, up to two soft-shaping constraints I groups of individual semidefinite shaping constraints	Herein
$M=L$, $0\le I\le 2$ $\boldsymbol{D}_{il}$ indefinite	L SINR constraints, up to two groups of individual indefinite shaping constraints	Huang-Palomar [5]
$M=L$, $\boldsymbol{D}_{il}$ indefinite, $i=1,2$ $\boldsymbol{D}_{il}$ semidefinite, $i\ge 3$	L SINR constraints, up to two groups of individual indefinite shaping constraints $I-2$ groups of individual semidefinite shaping constraints	Herein

decomposition of $\boldsymbol{X}$, (synthetically denoted as $\mathcal{D}(\boldsymbol{X},\boldsymbol{A}_1,\boldsymbol{A}_2)$), $\boldsymbol{X}=\sum_{r=1}^{R}\boldsymbol{x}_r\boldsymbol{x}_r^H$ such that

$$\boldsymbol{x}_r^H\boldsymbol{A}_1\boldsymbol{x}_r=\frac{\boldsymbol{X}\bullet\boldsymbol{A}_1}{R}\quad\text{and}\quad\boldsymbol{x}_r^H\boldsymbol{A}_2\boldsymbol{x}_r=\frac{\boldsymbol{X}\bullet\boldsymbol{A}_2}{R},$$

$r=1,\ldots,R$.

Let $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ and $(y_1^\star,\ldots,y_L^\star,z_{11}^\star,\ldots,z_{IL}^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ be optimal solutions of (P2) and (D2) respectively, and let $R_l=\operatorname{rank}(\boldsymbol{X}_l^\star)$ and $D_l=\dim(\operatorname{Null}(\boldsymbol{Z}_l))$, $l=1,\ldots,L$.

It follows by Lemma 5.3 that we can find a rank-one decomposition $\boldsymbol{X}_l^\star=\sum_{k=1}^{R_l}\boldsymbol{u}_{lk}\boldsymbol{u}_{lk}^H$ for each l such that

$$\boldsymbol{u}_{lk}^H\boldsymbol{D}_{il}\boldsymbol{u}_{lk}=\frac{\boldsymbol{D}_{il}\bullet\boldsymbol{X}_l^\star}{R_l},\ i=1,2,\ k=1,\ldots,R_l.$$

Observe that $\boldsymbol{u}_{lk}\in\operatorname{Range}(\boldsymbol{X}_l^\star)\subseteq\operatorname{Null}(\boldsymbol{Z}_l^\star),\forall k,l$. Then, take $\boldsymbol{u}_l=\boldsymbol{u}_{l1}(\in\operatorname{Null}(\boldsymbol{Z}_l^\star))$, $l=1,\ldots,L$, and formulate linear program (LP1) and its dual (DLP1), as displayed in (50) and (51) respectively. We claim problem (P2) has some properties similar to those in Theorem 5.2 with parameters satisfying (32), (33), and (52).

Theorem 5.4: Suppose that the parameters of (P2) comply with (32), (33) and (52). Suppose that both (P2) and (D2) are solvable, and let $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ and $(y_1^\star,\ldots,y_L^\star,z_{11}^\star,\ldots,z_{IL}^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$ be optimal solutions of (P2) and (D2) respectively. Perform the rank-one decomposition $\mathcal{D}(\boldsymbol{X}_l^\star,\boldsymbol{D}_{1l},\boldsymbol{D}_{2l})$ for each l, yielding $\boldsymbol{U}_l=[\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lR_l}]$, where $R_l=\operatorname{rank}(\boldsymbol{X}_l^\star)$, so that $\boldsymbol{X}_l^\star=\boldsymbol{U}_l\boldsymbol{U}_l^H$; take vectors $\boldsymbol{u}_l$ from $\{\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lR_l}\}$, $l=1,\ldots,L$, and formulate the LPs (LP1) and (DLP1). Then,

(i) (LP1) is bounded below, and $(y_1^\star,\ldots,y_L^\star)$ is feasible for (LP1) (i.e., (LP1) and (DLP1) are solvable).

Let $(x_1^\star,\ldots,x_L^\star)$ be an optimal solution of (DLP1). Suppose that $\boldsymbol{C}_l$, $l=1,\ldots,L$, fulfill (34). Then,

(i) $\sum_{l=1}^{L}\boldsymbol{u}_l^H\boldsymbol{A}_{ml}\boldsymbol{u}_l x_l^\star=b_m$, $m=1,\ldots,L$, and $x_l^\star>0$, $l=1,\ldots,L$, and $\boldsymbol{x}^\star=\boldsymbol{A}^{-1}\boldsymbol{b}$, where $\boldsymbol{A}$ is defined in (45);

(ii) $v(\text{D2})=v(\text{LP1})=v(\text{DLP1})=v(\text{P2})$, and $(x_1^\star\boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star\boldsymbol{u}_L\boldsymbol{u}_L^H)$ is an optimal solution of (P2).

Proof: See Appendix I. ■

Algorithm 5 summarizes the procedure to produce a rank-one optimal solution of (P2):

Algorithm 5: Procedure for Rank-One Solution of Separable SDP Problem With Individual Indefinite Shaping Constraints

`Input` : $\boldsymbol{C}_l$, $\boldsymbol{A}_{ml}$, b_m, $\boldsymbol{D}_{il}$, $i=1,\ldots,I$, $l=1,\ldots,L$, $m=1,\ldots,L$, satisfying **(32)–(34)** and **(52)**;

`Output` : an optimal solution $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$ with $\operatorname{rank}(\boldsymbol{X}_l^\star)=1,\forall l$;

1:solve SDPs (P2) and (D2), finding solutions $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_L^\star)$, and $(y_1^\star,\ldots,y_L^\star,z_{11}^\star,\ldots,z_{IL}^\star,\boldsymbol{Z}_1^\star,\ldots,\boldsymbol{Z}_L^\star)$;

2:perform the rank-one decompositions $\mathcal{D}(\boldsymbol{X}_l^\star,\boldsymbol{D}_{1l},\boldsymbol{D}_{2l}),\forall l$, outputting $\boldsymbol{X}_l^\star=\boldsymbol{U}_l\boldsymbol{U}_l^H$, $\boldsymbol{U}_l=[\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lR_l}]$, $l=1,\ldots,L$, where $R_l=\operatorname{rank}(\boldsymbol{X}_l^\star)$;

3:take vectors $\boldsymbol{u}_l\in\{\boldsymbol{u}_{l1},\ldots,\boldsymbol{u}_{lR_l}\}$, $l=1,\ldots,L$;

4:form the matrix $\boldsymbol{A}$ as in **(45)**, and compute $\boldsymbol{x}^\star=\boldsymbol{A}^{-1}\boldsymbol{b}$;

5:output $\boldsymbol{X}_l^\star=x_l^\star\boldsymbol{u}_l\boldsymbol{u}_l^H$, $l=1,\ldots,L$.

Last, we mention that it is possible to generate a rank-constrained solution of (P2) with a given rank profile $(P_1,\ldots,P_L)$ using Algorithm 3, but with the difference that in Step 1 of Algorithm 3 one has to solve (D2) and (P2) and additionally implement the specific rank-one decompositions.

VI. SUMMARY OF SOLVABLE INSTANCES OF OBP

In this section, we summarize all known solvable instances of the general QCQP downlink (unicast) beamforming problem (OBP) (see Table I), as well as an account of the complexity of the algorithms.

We remark that when problem (OBP) has soft-shaping constraints (in additional to SINR constraints and individual shaping constraints), only the primal method (cf. Algorithm 1) can be employed (the dual-based Algorithms 2, 4, and 5 cannot be used). When the problem has no soft-shaping constraints, the dual-based method is preferred due to its lower computational complexity as elaborated next.

We now compare the computational complexity of the primal and dual methods when solving the beamforming problem (OBP) with only SINR constraints. The primal method consists of solving the separable SDP, which has a worst-case complexity of $O\left((L^3K^{3.5}+L^4)\log 1/\eta\right)$, where η is the desired accuracy of the solution (cf. [16]), and the iterative rank reduction procedure, each step of which contains a eigenvalue decomposition that requires $O\left((\max_{1\le l\le L}\{R_l\})^3\right)$ flops and finding the null space of $\boldsymbol{B}\in\mathbb{R}^{L\times U}$ (i.e., solving the system of linear equations $\boldsymbol{Bx}=\boldsymbol{0}$) that requires $O\left(U^3\right)$ flops with $U=\sum_{l=1}^{L}R_l^2$. The dual method requires solving the same separable SDP and finding the vectors $\boldsymbol{u}_l$ in the respective null spaces and computing the closed-form solution of the LP which involves $O\left(L^3\right)$ flops. As to finding $\boldsymbol{u}_l$'s, it can be done very efficiently by simply computing the eigenvector corresponding to the maximum eigenvalue of $\boldsymbol{X}_l^\star$ (instead of the null space of the dual solution[4]) and that can be done efficiently with the power iteration method (see [18, pp. 330–332]), whose computational complexity is $O(K^2\log(1/\eta))$. Although the primal method has a higher complexity, it solves a separable SDP (outputting a rank-constrained solution) which has more flexibility on the parameter restrictions, e.g., all $\boldsymbol{C}_l$ do not have to be positive definite, the inequality directions of the global shaping constraints [cf. (4)] can be arbitrary, and parameters $\boldsymbol{T}_{ml}$ in the global shaping constraints can be any Hermitian matrix, and so forth.

VII. Numerical Examples

The present section is aimed at illustrating the effectiveness of the proposed downlink algorithms for the optimal beamforming problem. We consider a simulated scenario with a base station feeding signals simultaneously to three single-antenna users, i.e., $N=1$ and $L=3$ in problem (OBP). The users are placed at $\theta_1=-5°$, $\theta_2=10°$ and $\theta_3=25°$ relative to the array broadside of the base station. The channel covariance matrix for users $m=1,2,3$ is generated according to (see [3])

$$[\boldsymbol{R}_m(\theta_m)]_{pq}=e^{j\pi(p-q)\sin\theta_m}e^{-(-\pi(p-q)\sigma_\theta\cos\theta_m)^2/2} \quad (53)$$

$p,q\in\{1,\ldots,K\}$, where $\sigma_\theta=2°$ is the angular spread of local scatterers surrounding user m (as seen from the base station) and K represents the number of transmit antenna elements equipped in the base station. The noise variance is set $\sigma^2=0.1$ for each user. The SINR threshold value for all the three users is set to a common ρ. We make use of the optimization package CVX (see [19]) to solve the SDPs.

1) Simulation Example 1: In this example, we present simulation results when problem (OBP) has multiple null interference constraints beside the SINR constraints, and show how the total transmission power is affected by the number of null interference constraints. In addition to internal users, we consider six (i.e., $6=M-L$, $M=9$) external users belonging to other coexisting wireless systems, and they are located respectively at $\tilde{\theta}_m$, $m=L+1,\ldots,M$, relative to the array broadside of the base station with $K=8$ antenna elements. The channel between the base station and external user $\tilde{\theta}_m$ is given by (assuming a uniformly spaced array at the base station):

$$\boldsymbol{h}(\tilde{\theta}_m)=[1\ e^{j\phi_m}\ \ldots\ e^{j(K-1)\phi_m}]^T,\ m=4,\ldots,M \quad (54)$$

[4]It is not necessary to characterize the whole null space, whose cost would be higher.

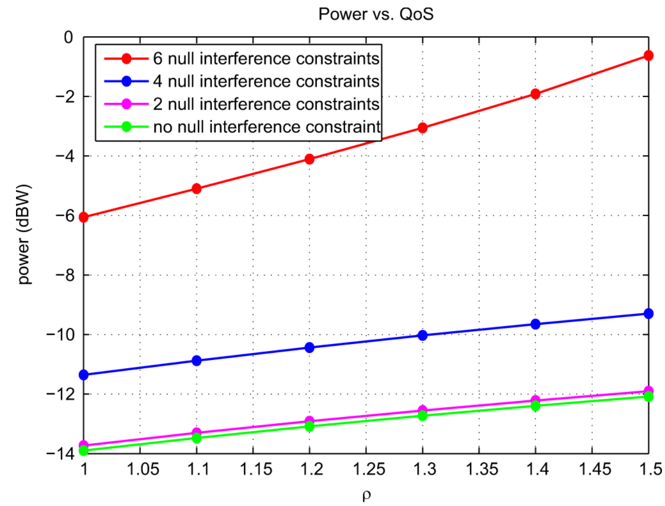

Fig. 2. Minimal transmission power versus the threshold of SINR, with different numbers of null interference constraint.

where $\phi_m=2\pi d\sin(\tilde{\theta}_m)/\lambda$, $d/\lambda=1/2$ (i.e., the antenna elements are spaced half a wavelength). This corresponds to problem (OBP) with $L=3$ SINR constraints and $M-L=6$ null interference constraints (or, equivalently, six groups of individual shaping constraints). Fig. 2 illustrates the minimal total transmission power versus the required SINR ρ for the cases of no null interference constraint, two null interference constraints (with $\tilde{\theta}_4=50°$ and $\tilde{\theta}_5=70°$), four null interference constraints (with $\tilde{\theta}_6=-35°$ and $\tilde{\theta}_7=35°$ in addition to $\tilde{\theta}_4$ and $\tilde{\theta}_5$), and six null interference constraints (with $\tilde{\theta}_8=-25°$ and $\tilde{\theta}_9=0°$ in addition to $\tilde{\theta}_m, m=4,5,6,7$). It can be seen from the figure that higher and higher total transmission power is required to satisfy null interference constraints for more and more external users, as well as the same SINR level ρ to the three internal users.

2) Simulation Example 2: This example shows the results for problem (OBP) where the base station is equipped with $K=12$ antenna elements and nine null interference constraints for nine external users at $\{-65°,-60°,-55°,-35°,-25°,-15°,35°,50°,70°\}$, together with the three SINR constraints, are involved. In order to illustrate the effect of the additional null interference constraints, we evaluate the power radiation pattern of the base station, for $\theta\in[-90°,90°]$, according to

$$P(\theta)=\boldsymbol{h}(\theta)\boldsymbol{h}(\theta)^H\bullet(\boldsymbol{w}_1^\star\boldsymbol{w}_1^{\star H}+\boldsymbol{w}_2^\star\boldsymbol{w}_2^{\star H}+\boldsymbol{w}_3^\star\boldsymbol{w}_3^{\star H}) \quad (55)$$

where $(\boldsymbol{w}_1^\star,\boldsymbol{w}_2^\star,\boldsymbol{w}_3^\star)$ is a triple of optimal beamvectors, and $\boldsymbol{h}(\theta)$ is defined in (54). Fig. 3 displays the radiation pattern of the base station with the SINR threshold value $\rho=1$ (the minimal required transmission power is 15.95 dBm).

3) Simulation Example 3: In this example, we consider the robust null-shaping interference constraint [cf. (8)] in the direction of an external user (say, located at $\tilde{\theta}_m$), and this can be realized by adding two null interference constraints:

$$\boldsymbol{h}(\tilde{\theta}_m)\boldsymbol{h}(\tilde{\theta}_m)^H\bullet(\boldsymbol{w}_1\boldsymbol{w}_1^H+\boldsymbol{w}_2\boldsymbol{w}_2^H+\boldsymbol{w}_3\boldsymbol{w}_3^H)\le 0, \quad (56)$$

and

$$\boldsymbol{g}(\tilde{\theta}_m)\boldsymbol{g}(\tilde{\theta}_m)^H\bullet(\boldsymbol{w}_1\boldsymbol{w}_1^H+\boldsymbol{w}_2\boldsymbol{w}_2^H+\boldsymbol{w}_3\boldsymbol{w}_3^H)\le 0 \quad (57)$$

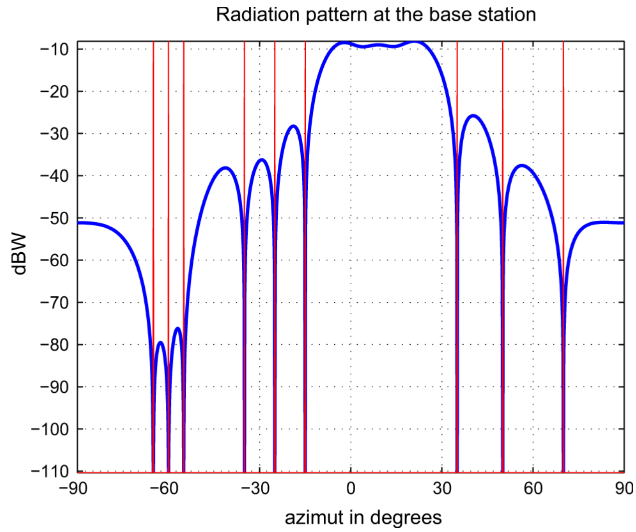

Fig. 3. Radiation pattern of the base station, for the problem with three SINR constraints and nine null-shaping interference constraints and $K = 12$. The required transmit power is 15.95 dBm.

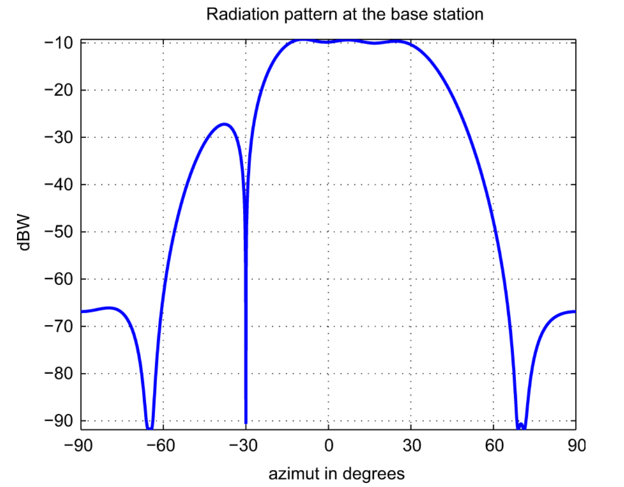

Fig. 4. Radiation pattern of the base station, for the problem with three SINR constraints, one null interference constraint, two robust null interference constraint, and $K = 8$. The required transmit power is 16.94 dBm.

where

$$\boldsymbol{g}(\tilde{\theta}_m) = \frac{d\boldsymbol{h}_{(\tilde{\theta}_m)}}{d\tilde{\theta}_m} = [0 \quad j\psi_m e^{j\phi_m} \quad \cdots \quad j(K-1)\psi_m e^{j(K-1)\phi_m}]^T \tag{58}$$

and $\psi_m = 2\pi d\cos(\tilde{\theta}_m)/\lambda$ and ϕ_m is the same as the one in (54). To better control the robust region of the null-shaping interference around external user $\tilde{\theta}_m$, we introduce two more null interference constraints for the external user

$$\boldsymbol{h}_m(\theta)\boldsymbol{h}_m(\theta)^H \bullet (\boldsymbol{w}_1\boldsymbol{w}_1^H + \boldsymbol{w}_2\boldsymbol{w}_2^H + \boldsymbol{w}_3\boldsymbol{w}_3^H) \leq 0, \tag{59}$$

for $\theta = \tilde{\theta}_m - 1^\circ, \tilde{\theta}_m + 1^\circ$. In other words, there are four constraints [i.e., (56), (57), and (59)] describing a robust null-shaping interference constraint around user $\tilde{\theta}_m$. Assume that the base station has $K = 8$ transmit antennas, and three external users located, respectively, at $\{-65^\circ, -30^\circ, 70^\circ\}$ are considered beside the three internal users. For external user $\tilde{\theta}_m = -30^\circ$, we impose only one null interference constraint on it, while for external users $\tilde{\theta}_1 = -65^\circ$ and $\tilde{\theta}_1 = 70^\circ$, we impose one robust null interference constraint on each of them. Therefore, the resulting optimal beamforming problem has three SINR constraints and nine null interference constraints. Fig. 4 depicts the radiation pattern of the base station (the required minimal transmission power is 16.94 dBm). As expected, the power radiated over $[-66^\circ, -64^\circ]$ and $[69^\circ, 71^\circ]$ is sufficiently lower so that it is almost negligible.

VIII. Conclusion

In this paper, we have considered the optimal beamforming problem minimizing the transmission power subject to SINR constraints, global and individual shaping interference constraints (e.g., to protect other users from coexisting systems). Although the problem belongs to the class of separable homogeneous QCQP which is typically hard to solve in general, we have proposed efficient algorithms for the beamforming problem. The presented algorithms mainly consist of solving the SDP relaxation of the problem, formulating an LP by using the properties of optimal solution of the dual SDP, and solving the LP (with closed-form solution) to find a rank-one optimal solution of the separable SDP. The proof is based on the LP strong duality. The resulting algorithms can output a rank-one solution, as well as a rank-constrained solution with a prefixed rank profile. Based on these, we have identified the subclasses of the optimal beamforming problem that are "hidden" convex, in the sense that the corresponding SDP relaxation has always a rank-one optimal solution.

Appendix

A. Proof of Theorem 3.1

Proof: The proof is just slightly different from that of [5, Lemma 3.1] (see [5, pp. 675–676]) when checking the primal feasibility and complementarity (optimality) in each iteration of the purification process (rank reduction procedure). Precisely, we need to additionally verify $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l' \trianglerighteq_{il} 0$, $\forall i, l$, and $z_{il}^\star \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l' = 0$, $\forall l \notin \mathcal{E}_i$, $i = 1, \ldots, L$, where $\boldsymbol{X}_l' = \boldsymbol{V}_l(\boldsymbol{I} - (1/\delta_{l_0 k_0})\boldsymbol{\Delta}_l)\boldsymbol{V}_l^H$, $\boldsymbol{X}_l^\star = \boldsymbol{V}_l\boldsymbol{V}_l^H$ and $\boldsymbol{I} - (1/\delta_{l_0 k_0})\boldsymbol{\Delta}_l \succeq \boldsymbol{0}$. Indeed, if $l \in \mathcal{E}_i$, then $\boldsymbol{V}_l^H\boldsymbol{D}_{il}\boldsymbol{V}_l = \boldsymbol{0}$, and $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l' = \boldsymbol{V}_l^H\boldsymbol{D}_{il}\boldsymbol{V}_l \bullet (\boldsymbol{I} - (1/\delta_{l_0 k_0})\boldsymbol{\Delta}_l) = 0$; if $l \notin \mathcal{E}_i$, then it is evident that $\boldsymbol{D}_{il} \bullet \boldsymbol{X}_l' \geq 0$. Since all columns of $\boldsymbol{V}_l$ are in $\mathrm{Range}(\boldsymbol{X}_l^\star)$, hence $z_{il}^\star \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l' = 0$, $\forall l \notin \mathcal{E}_i$. ■

B. Proof of Proposition 4.1

Proof: (i) First of all, we show $\boldsymbol{X}_l^\star \neq \boldsymbol{0}, \forall l$. In fact, suppose that $\boldsymbol{X}_1^\star = \boldsymbol{0}$, then one has the contradiction

$$0 \trianglerighteq_1 b_1 - \sum_{l=2}^{L} \boldsymbol{A}_{1l} \bullet \boldsymbol{X}_l^\star > 0$$

due to the parameter conditions (32) and (33).

(ii) Now, we show $\boldsymbol{Z}_l^\star \succeq \boldsymbol{0}$, and $\nsucc \boldsymbol{0}$, $l = 1, \ldots, L$. Suppose $\boldsymbol{Z}_l^\star \succ \boldsymbol{0}$, for some $l \in \{1, \ldots, L\}$, say $\boldsymbol{Z}_1^\star \succ \boldsymbol{0}$; then $\boldsymbol{X}_1^\star = \boldsymbol{0}$, from the complementary condition (30). However, this is not possible since we have shown the fact that $\boldsymbol{X}_l^\star \neq \boldsymbol{0}$, $\forall l$. ■

C. Proof of Proposition 4.2

Proof: Since $\trianglerighteq_m \in \{\geq, =\}$, $m = 1, \ldots, L$, and $\trianglerighteq_m \in \{\leq\}$, $m = L+1, \ldots, M$, hence, $y_m^\star \geq 0$ for those m with $\trianglerighteq_m \in \{\geq\}$, $y_m^\star \in \mathbb{R}$ for those m with $\trianglerighteq_m \in \{=\}$, and $y_m^\star \leq 0$ for $m \in \{L+1, \ldots, M\}$. In order to show $y_m^\star > 0, m = 1, \ldots, L$, we shall first prove the fact that $y_m^\star \geq 0$ for $m = 1, \ldots, L$ (or more precisely, for $m : \trianglerighteq_m \in \{=\}$).

Assume, without loss of generality, that $y_1^\star \geq 0, \ldots, y_{L_0}^\star \geq 0$, $y_{L_0+1}^\star < 0, \ldots, y_L^\star < 0$, for some $L_0 \in \{1, \ldots, L-1\}$. Then, $(y_1^\star, \ldots, y_{L_0}^\star, 0, \ldots, 0, y_{L+1}^\star, \ldots, y_M^\star)$, together with $\boldsymbol{Z}_l' = \boldsymbol{C}_l - \sum_{m=1}^{L_0} y_m^\star \boldsymbol{A}_{ml} - \sum_{m=L+1}^{M} y_m^\star \boldsymbol{A}_{ml}$, $l = 1, \ldots, L$, is feasible for problem (D), and has the objective function value $\sum_{m=1}^{L_0} b_m y_m^\star + \sum_{m=L+1}^{M} b_m y_m^\star > \sum_{m=1}^{M} b_m y_m^\star$. Thus $(y_1^\star, \ldots, y_M^\star)$ cannot be optimal, which is a contradiction. Therefore, we have $y_m^\star \geq 0$, $m = 1, \ldots, L$. Let us check the feasibility of $(y_1^\star, \ldots, y_{L_0}^\star, 0, \ldots, 0, y_{L+1}^\star, \ldots, y_M^\star)$. Indeed, for $l = 1, \ldots, L_0$, one has $\boldsymbol{Z}_l' = \boldsymbol{C}_l - \sum_{m=1}^{L_0} y_m^\star \boldsymbol{A}_{ml} - \sum_{m=L+1}^{M} y_m^\star \boldsymbol{A}_{ml} \succeq \sum_{m=L_0+1}^{L} y_m^\star \boldsymbol{A}_{ml} \succeq \boldsymbol{0}$, where the first $\succeq$ relation is due to $\boldsymbol{Z}_l \succeq \boldsymbol{0}$, and the second $\succeq$ is due to the assumption that $y_m^\star < 0$, $-\boldsymbol{A}_{ml} \succeq 0$, for $m = L_0+1, \ldots, L$, $l = 1, \ldots, L_0$. For $l = L_0+1, \ldots, L$, one also has $\boldsymbol{Z}_l' = \boldsymbol{C}_l - \sum_{m=1}^{L_0} y_m^\star \boldsymbol{A}_{ml} - \sum_{m=L+1}^{M} y_m^\star \boldsymbol{A}_{ml} \succeq \boldsymbol{0}$, due to $y_m^\star \geq 0$ for $m = 1, \ldots, L_0$, $y_m^\star \leq 0$ for $m = L+1, \ldots, M$, (32), (34), and (35).

Now we wish to prove $y_1^\star > 0, \ldots, y_L^\star > 0$. Suppose that one of $y_l^\star = 0$, say $y_1^\star = 0$, then $\boldsymbol{Z}_1^\star = \boldsymbol{C}_1 - \sum_{m=1}^{M} y_m^\star \boldsymbol{A}_{m1} = \boldsymbol{C}_1 - \sum_{m=2}^{L} y_m^\star \boldsymbol{A}_{m1} - \sum_{m=L+1}^{M} y_m^\star \boldsymbol{A}_{m1} \succ \boldsymbol{0}$, which is a contradiction to $\boldsymbol{Z}_1^\star \not\succ \boldsymbol{0}$ that we have shown in Proposition 4.1.

It follows from (31) that $\sum_{l=1}^{L} \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star = b_m$, $m = 1, \ldots, L$. ∎

D. Proof of Proposition 4.3

Proof: Since $(y_{L+1}, \ldots, y_M) \in -\mathbb{R}_+^{M-L}$, $(y_1, \ldots, y_L) \in \mathcal{Y}(y_{L+1}, \ldots, y_M)$ and the assumptions (32), (35), (34) are valid, hence, it follows from (36) that each $\boldsymbol{A}_{ll}$ has at least one positive eigenvalue.

(i) Suppose $(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_L)$ is defined by the point $(y_1, \ldots, y_M)$ as in set $\mathcal{Y}(y_{L+1}, \ldots, y_M)$, and let

$$\boldsymbol{E}_l = \boldsymbol{C}_l - \sum_{m=1,\, m\neq l}^{M} y_m \boldsymbol{A}_{ml},\ l = 1, \ldots, L.$$

Clearly, $\boldsymbol{E}_l \succ \boldsymbol{0}$, $l = 1, \ldots, L$. Then, we have

$$\begin{aligned}\boldsymbol{Z}_l &= \boldsymbol{C}_l - \sum_{m=1,\, m\neq l}^{M} y_m \boldsymbol{A}_{ml} - y_l \boldsymbol{A}_{ll} \\ &= \boldsymbol{E}_l - y_l \boldsymbol{A}_{ll} \\ &= \boldsymbol{E}_l^{1/2}(\boldsymbol{I} - y_l \boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2}) \boldsymbol{E}_l^{1/2}. \end{aligned} \tag{60}$$

Recall that each $\boldsymbol{A}_{ll}$, $l \in \{1, \ldots, L\}$, has at least one positive eigenvalue, thus $\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2}) > 0$, $l = 1, \ldots, L$. From (60), it follows that $0 \leq y_l < 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2})$ leads to $\boldsymbol{Z}_l \succ \boldsymbol{0}$, for $l = 1, \ldots, L$ (for either the case that $\boldsymbol{A}_{ll}$ is positive semidefinite and nonzero, or the case that $\boldsymbol{A}_{ll}$ has both positive and negative eigenvalues). Since $\boldsymbol{Z}_l \not\succ \boldsymbol{0}$ (from the definition of $\mathcal{Y}(y_{L+1}, \ldots, y_M)$), then $y_l \geq 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2})$, $l = 1, \ldots, L$.

(ii) From (60), we see that $y_l = 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2})$ implies $\boldsymbol{Z}_l \succeq \boldsymbol{0}$ and $\not\succ \boldsymbol{0}$, $l = 1, \ldots, L$, and that $0 \leq y_l < 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2})$ implies $\boldsymbol{Z}_l \succ \boldsymbol{0}$, $l = 1, \ldots, L$, and that $y_l > 1/\lambda_{\max}(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2})$ implies $\boldsymbol{Z}_l \not\succeq \boldsymbol{0}$, $l = 1, \ldots, L$.

(iii) Suppose that there exists $(y_1', \ldots, y_L') \in \mathcal{Y}(y_{L+1}, \ldots, y_M)$ such that $(y_1', \ldots, y_L') \neq (y_1, \ldots, y_L)$. Then it follows from (i) that $y_m' > 0$, $m = 1, \ldots, L$. Let $\lambda = \max_{1\leq m\leq L} y_m/y_m'$, and suppose that $\lambda > 1$ without loss of generality (otherwise let $\lambda = \max_{1\leq m\leq L} y_m'/y_m > 1$). Also suppose that $\lambda = y_1/y_1' \geq y_m/y_m'$, $m = 2, \ldots, L$. It follows that $\lambda y_1' \boldsymbol{A}_{11} = y_1 \boldsymbol{A}_{11} \preceq \boldsymbol{C}_1 - \sum_{m=2}^{L} y_m \boldsymbol{A}_{m1} - \sum_{m=L+1}^{M} y_m \boldsymbol{A}_{m1} \preceq \boldsymbol{C}_1 - \sum_{m=2}^{L} \lambda y_m' \boldsymbol{A}_{m1} - \sum_{m=L+1}^{M} \lambda y_m \boldsymbol{A}_{m1} \prec \lambda(\boldsymbol{C}_1 - \sum_{m=2}^{L} y_m' \boldsymbol{A}_{m1} - \sum_{m=L+1}^{M} y_m \boldsymbol{A}_{m1})$, which would imply $\boldsymbol{Z}_1' = \boldsymbol{C}_1 - \sum_{m=1}^{L} y_m' A_{m1} - \sum_{m=L+1}^{M} y_m A_{m1} \succ \boldsymbol{0}$. But this is not possible since $(y_1', \ldots, y_L') \in \mathcal{Y}(y_{L+1}, \ldots, y_M)$.

It follows from the proof of (ii) that the set $\mathcal{Y}(y_{L+1}, \ldots, y_M)$ is a singleton and contains the point: See the equation at the bottom of the page. ∎

E. Proof of Theorem 4.4

Proof: (i) Since both (LP1) and (DLP1) are solvable, we suppose that $(\hat{y}_1, \ldots, \hat{y}_L)$ is an optimal solution of (LP1). Thus, $(x_1^\star, \ldots, x_L^\star)$ and $(\hat{y}_1, \ldots, \hat{y}_L)$ satisfy the complementary conditions (43) and (44).

To proceed, we claim that any feasible point $(y_1, \ldots, y_L)$ of (LP1) must be positive, i.e., $y_m > 0$, $m = 1, \ldots, L$. To see this, assume that $y_1 = 0$, then by the parameter assumptions, one has the contradiction

$$0 < \boldsymbol{u}_1^H \boldsymbol{C}_1 \boldsymbol{u}_1 \leq \sum_{m=2}^{L} y_m \boldsymbol{u}_1^H \boldsymbol{A}_{m1} \boldsymbol{u}_1 \leq 0.$$

Consequently, every y_m must be positive. Now, from (44), this implies that $\sum_{l=1}^{L} \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l x_l^\star = b_m$, $m = 1, \ldots, L$.

(ii) Note that each $\boldsymbol{A}_{ll}$ has at least one positive eigenvalue and $\boldsymbol{u}_l^H \boldsymbol{A}_{ll} \boldsymbol{u}_l > 0$, $\forall l$, from (41). Indeed, if some $\boldsymbol{A}_{ll}$ is negative semidefinite, then problem (P1) is infeasible, which contradicts the assumption that (P1) is solvable. Also $\boldsymbol{u}_l^H \boldsymbol{A}_{ll} \boldsymbol{u}_l =$

$$\begin{aligned} y_l &= \frac{1}{\lambda_{\max}\left(\boldsymbol{E}_l^{-1/2} \boldsymbol{A}_{ll} \boldsymbol{E}_l^{-1/2}\right)} \\ &= \frac{1}{\lambda_{\max}\left(\left(\boldsymbol{C}_l - \sum_{m\neq l} y_m \boldsymbol{A}_{ml}\right)^{-1/2} \boldsymbol{A}_{ll} \left(\boldsymbol{C}_l - \sum_{m\neq l} y_m \boldsymbol{A}_{ml}\right)^{-1/2}\right)}, \quad l = 1, \ldots, L. \end{aligned}$$

$1/y_l^\star(\boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1,\,m\neq l}^L y_m^\star \boldsymbol{u}_l^H \boldsymbol{A}_{ml}\boldsymbol{u}_l) > 0, \forall l$, since $\boldsymbol{u}_l \in \mathrm{Null}(\boldsymbol{Z}_l^\star)$ and $y_l^\star > 0$ (from Proposition 4.2). It follows that

$$x_m^\star = \frac{1}{\boldsymbol{u}_m^H \boldsymbol{A}_{mm}\boldsymbol{u}_m}\left(b_m - \sum_{l=1,l\neq m}^{L} \boldsymbol{u}_l^H \boldsymbol{A}_{ml}\boldsymbol{u}_l x_l^\star\right) > 0$$

$m = 1,\ldots,L$,, since $b_m > 0$ and $-\boldsymbol{u}_l^H \boldsymbol{A}_{ml}\boldsymbol{u}_l \geq 0$ $(m \neq l)$ from the parameter assumptions (32) and (33).

(iii) We claim that $\boldsymbol{x}$ satisfying (i) and (ii) must be unique, i.e., the solution for the system of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} > \boldsymbol{0}$ is unique. Suppose that $\hat{\boldsymbol{x}}(\neq \boldsymbol{x})$ is another solution for the system, and let $\lambda = \max_{1\leq l\leq L} x_l/\hat{x}_l$. Without loss of generality, we assume that $\lambda > 1$ and $x_1/\hat{x}_1 = \lambda$; thus we have $\lambda = x_1/\hat{x}_1 \geq x_l/\hat{x}_l$, $l = 2,\ldots,L$, which implies that $\lambda\hat{x}_1 = x_1$ and $\lambda\hat{x}_l \geq x_l$, $l = 2,\ldots,L$. Then one gets the contradiction $b_1 = \sum_{l=1}^L \boldsymbol{u}_l^H \boldsymbol{A}_{1l}\boldsymbol{u}_l x_l = \lambda \boldsymbol{u}_1^H \boldsymbol{A}_{11}\boldsymbol{u}_1 \hat{x}_1 + \sum_{l=2}^L \boldsymbol{u}_l^H \boldsymbol{A}_{1l}\boldsymbol{u}_l x_l \geq \lambda\left(\boldsymbol{u}_1^H \boldsymbol{A}_{11}\boldsymbol{u}_1 \hat{x}_1 + \sum_{l=2}^L \boldsymbol{u}_l^H \boldsymbol{A}_{1l}\boldsymbol{u}_l \hat{x}_l\right) = \lambda b_1 > b_1$.

Suppose that the square matrix $\boldsymbol{A}$ is singular, then there is a nonzero $\boldsymbol{x}_0$ such that $\boldsymbol{A}\boldsymbol{x}_0 = \boldsymbol{0}$, and since $\boldsymbol{x} > \boldsymbol{0}$, we have that for η sufficiently small $\boldsymbol{x}+\eta\boldsymbol{x}_0 > \boldsymbol{0}$. This implies that the vector $\boldsymbol{x}+\eta\boldsymbol{x}_0$ is a solution for the system of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} > \boldsymbol{0}$, which is not possible since the system has only one solution. Therefore, $\boldsymbol{A}$ is invertible, which yields $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$. If $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \not> \boldsymbol{0}$, then (DLP1) has no optimal solution, which is contradictory to the solvability of (DLP1).

(iv) It suffices to prove the inequality chain $v(\mathrm{D1}) \geq v(\mathrm{LP1}) \geq v(\mathrm{DLP1}) \geq v(\mathrm{P1}) \geq v(\mathrm{D1})$. It is clear that $v(\mathrm{LP1}) = v(\mathrm{DLP1})$ and $v(\mathrm{P1}) = v(\mathrm{D1})$, due to strong duality. Notice that the optimal solution $(y_1^\star,\ldots,y_L^\star)$ of (D1) is also feasible for (LP1), and that (D1) and (LP1) have the same objective function, then we can assert that $v(\mathrm{D1}) \geq v(\mathrm{LP1})$. Likewise, observing that $(x_1^\star \boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star \boldsymbol{u}_L\boldsymbol{u}_L^H)$ is feasible for (P1), we conclude that $v(\mathrm{DLP1}) \geq v(\mathrm{P1})$. Therefore, we arrive at $v(\mathrm{D1}) = v(\mathrm{LP1}) = v(\mathrm{DLP1}) = v(\mathrm{P1})$. Further, it is easily verified that $(y_1^\star,\ldots,y_L^\star)$ is optimal for (LP1) and $(x_1^\star \boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star \boldsymbol{u}_L\boldsymbol{u}_L^H)$ is optimal for (P1). ■

F. Proof of Theorem 4.5

Proof: This proof is similar to that of Theorem 4.4.

(i) Assume that $(\hat{y}_1,\ldots,\hat{y}_L)$ is an optimal solution of (LP2). Thus, $(\hat{y}_1,\ldots,\hat{y}_L)$, together with the solution $\{x_{lk}^\star\}$, complies with the complementary conditions of (LP2) and (DLP2)

$$x_{lk}^\star\left(\boldsymbol{u}_{lk}^H \boldsymbol{C}_l \boldsymbol{u}_{lk} - \sum_{m=1}^L \hat{y}_m \boldsymbol{u}_{lk}^H \boldsymbol{A}_{ml}\boldsymbol{u}_{lk}\right) = 0 \tag{61}$$

$l = 1,\ldots,L$, $k = 1,\ldots,P_l$, and

$$\hat{y}_m\left(\sum_{l=1}^L\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H \boldsymbol{A}_{ml}\boldsymbol{u}_{lk} x_{lk}^\star - b_m\right) = 0,\ m = 1,\ldots,L. \tag{62}$$

It is easily seen from (46) that any feasible point $(y_1,\ldots,y_L)$ of (LP2) is positive, i.e., $y_l > 0, \forall l$ (due to a similar argument to the second paragraph of the proof of Theorem 4.4). Thus $\hat{y}_l > 0$, $\forall l$. It follows from (62) that

$$\sum_{l=1}^L\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H \boldsymbol{A}_{ml}\boldsymbol{u}_{lk} x_{lk}^\star = b_m,\ m = 1,\ldots,L.$$

(ii) Suppose that $x_{1k}^\star$, $k = 1,\ldots,P_1$, are zero. It follows from (i) that $\sum_{l=1}^L\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H \boldsymbol{A}_{1l}\boldsymbol{u}_{lk} x_{lk}^\star = \sum_{l=2}^L\sum_{k=1}^{P_l} \boldsymbol{u}_{lk}^H \boldsymbol{A}_{1l}\ \boldsymbol{u}_{lk} x_{lk}^\star = b_1$, which is not true since the left-hand side (LHS) is nonpositive and the right-hand side (RHS) is positive. We thus conclude that at least one element of $\{x_{l1}^\star,\ldots,x_{lP_l}^\star\}$ is positive, $l = 1,\ldots,L$.

(iii) The proof is similar to the proof of (iii) of Theorem 4.4, and thus omitted. ■

G. Proof of Proposition 5.1

Proof: Applying Proposition 4.1 (with $M = (I+1)\times L$) to (P2) and (D2), one concludes that (i) and (ii) hold. It follows from (18)–(20) that $y_m^\star > 0$ and $\sum_{l=1}^L \boldsymbol{A}_{ml} \bullet \boldsymbol{X}_l^\star = b_m$, $m = 1,\ldots,L$. Indeed, from (18) and (20), we have

$$\boldsymbol{Z}_l^\star \bullet \boldsymbol{X}_l^\star = \left(\boldsymbol{C}_l - \sum_{m=1}^L y_m^\star \boldsymbol{A}_{ml} - \sum_{i=1}^I z_{il}^\star \boldsymbol{D}_{il}\right) \bullet \boldsymbol{X}_l^\star = \left(\boldsymbol{C}_l - \sum_{m=1}^L y_m^\star \boldsymbol{A}_{ml}\right) \bullet \boldsymbol{X}_l^\star = 0,\ \forall l$$

which means that $y_m^\star$ cannot be zero for any $m \in \{1,\ldots,L\}$ (otherwise, say $y_1^\star = 0$, then $\boldsymbol{C}_1 - \sum_{m=2}^L y_m^\star \boldsymbol{A}_{m1} \succ \boldsymbol{0}$ and $\boldsymbol{X}_1^\star = \boldsymbol{0}$ which is not possible). ■

H. Proof of Theorem 5.2

Proof: (i) It is easily seen that (LP1) is bounded below, due to the constraints $y_m \geq 0$ and the assumptions $b_m > 0$. Now, we wish to show that the solution $(y_1^\star,\ldots,y_L^\star)$ of (D2) is feasible for (LP1).

Since $\boldsymbol{u}_l \in \mathrm{Range}(\boldsymbol{X}_l^\star) \subseteq \mathrm{Null}(\boldsymbol{Z}_l^\star)$, $\forall l$, hence $0 = \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1}^L y_m^\star \boldsymbol{u}_l^H \boldsymbol{A}_{ml}\boldsymbol{u}_l - \sum_{i=1}^I z_{il}^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il}\boldsymbol{u}_l$. Combining the complementary condition in (20) $z_{il}^\star \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l^\star = 0$ with the fact that $\boldsymbol{u}_l \in \mathrm{Range}(\boldsymbol{X}_l^\star)$ yields

$$z_{il}^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il}\boldsymbol{u}_l = 0,\ i = 1,\ldots,I,\ l = 1,\ldots,L. \tag{63}$$

Therefore, $0 = \boldsymbol{u}_l^H \boldsymbol{Z}_l \boldsymbol{u}_l = \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1}^L y_m^\star \boldsymbol{u}_l^H \boldsymbol{A}_{ml}\boldsymbol{u}_l$, and accordingly $(y_1^\star,\ldots,y_L^\star)$ is feasible for (LP1).

Since (LP1) is bounded below and feasible, hence it follows by the strong duality theorem for LP that both (LP1) and (DLP1) are solvable.

(ii) The proof is completely similar to those of (i), (ii), and (iii) of Theorem 4.4, and we skip it here.

(iii) It is easily seen that $(x_1^\star \boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star \boldsymbol{u}_L\boldsymbol{u}_L^H)$ fulfills the individual shaping constraints, that is, $\boldsymbol{D}_{il} \bullet (x_l^\star \boldsymbol{u}_l\boldsymbol{u}_l^H) = (\geq) 0$, $\forall l \in (\notin)\mathcal{E}_i$, $\forall i$. This, together with (ii), gives that $(x_1^\star \boldsymbol{u}_1\boldsymbol{u}_1^H,\ldots,x_L^\star \boldsymbol{u}_L\boldsymbol{u}_L^H)$ for feasible to (P2), hence, we have $v(\mathrm{DLP1}) \geq v(\mathrm{P2})$. Since the optimal solution $(y_1^\star,\ldots,y_L^\star)$ of (D2) is feasible for (LP1), hence, $v(\mathrm{D2}) \geq v(\mathrm{LP1})$. Thus, we arrive at the inequality chain

$v(\text{DLP1}) \geq v(\text{P2}) \geq v(\text{D2}) \geq v(\text{LP1}) \geq v(\text{DLP1})$, and it is easily seen that $(x_1^\star \boldsymbol{u}_1 \boldsymbol{u}_1^H, \ldots, x_L^\star \boldsymbol{u}_L \boldsymbol{u}_L^H)$ is optimal to (P2). ■

I. Proof of Theorem 5.4

Proof: (i) We show that $(y_1^\star, \ldots, y_L^\star)$ is feasible to (LP1). Observe that the vectors $\boldsymbol{u}_l \in \text{Range}(\boldsymbol{X}_l^\star) \subseteq \text{Null}(\boldsymbol{Z}_l^\star)$, $l = 1, \ldots, L$, have the property

$$\boldsymbol{u}_l^H \boldsymbol{D}_{1l} \boldsymbol{u}_l = \frac{\boldsymbol{D}_{1l} \bullet \boldsymbol{X}_l^\star}{R_l}, \; \boldsymbol{u}_l^H \boldsymbol{D}_{2l} \boldsymbol{u}_l = \frac{\boldsymbol{D}_{2l} \bullet \boldsymbol{X}_l^\star}{R_l}$$

$l = 1, \ldots, L$. It thus follows from the complementary condition (20) that $z_{il}^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il} \boldsymbol{u}_l = z_{il}^\star \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l^\star / R_l = 0$, $\forall l$, $i = 1, 2$, and from (63) that $z_{il}^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il} \boldsymbol{u}_l = 0$, $\forall l$, $i \geq 3$. Hence

$$\begin{aligned} 0 &= \boldsymbol{u}_l^H \boldsymbol{Z}_l^\star \boldsymbol{u}_l \\ &= \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1}^{L} y_m^\star \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l - \sum_{i=1}^{I} z_{il}^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il} \boldsymbol{u}_{il} \\ &= \boldsymbol{u}_l^H \boldsymbol{C}_l \boldsymbol{u}_l - \sum_{m=1}^{L} y_m^\star \boldsymbol{u}_l^H \boldsymbol{A}_{ml} \boldsymbol{u}_l, \; l = 1, \ldots, L. \end{aligned}$$

Therefore, $(y_1^\star, \ldots, y_L^\star)$ is feasible to (LP1). It is evident that (LP1) is bounded below.

(ii) The proof is the same as that of (ii) of Theorem 5.2.

(iii) From the proof of (iii) of Theorem 5.2, we know that $(x_1^\star \boldsymbol{u}_1 \boldsymbol{u}_1^H, \ldots, x_L^\star \boldsymbol{u}_L \boldsymbol{u}_L^H)$ satisfies the individual semidefinite shaping constraints, i.e., $\boldsymbol{D}_{il} \bullet (x_l^\star \boldsymbol{u}_l \boldsymbol{u}_l^H) = (\geq) 0$, $\forall l \in (\notin) \mathcal{E}_i$, $i \geq 3$. Further, the solution also satisfies the indefinite shaping constraints, i.e., $\boldsymbol{D}_{il} \bullet (x_l^\star \boldsymbol{u}_l \boldsymbol{u}_l^H) = (\geq) 0$, $\forall l \in (\notin) \mathcal{E}_i$, $i = 1, 2$. As a matter of fact, $\boldsymbol{D}_{il} \bullet (x_l^\star \boldsymbol{u}_l \boldsymbol{u}_l^H) = x_l^\star \boldsymbol{u}_l^H \boldsymbol{D}_{il} \boldsymbol{u}_l =$

$$\frac{x_l^\star \boldsymbol{D}_{il} \bullet \boldsymbol{X}_l^\star}{R_l} = \begin{cases} = 0, & \forall l \in \mathcal{E}_i \\ \geq 0, & \forall l \notin \mathcal{E}_i, \end{cases} \; i = 1, 2.$$

This combining with (ii) leads to that $(x_1^\star \boldsymbol{u}_1 \boldsymbol{u}_1^H, \ldots, x_L^\star \boldsymbol{u}_L \boldsymbol{u}_L^H)$ is feasible for (P2). By the same argument as the last paragraph of the proof of Theorem 5.2, we conclude that $v(\text{DLP1}) \geq v(\text{P2}) \geq v(\text{D2}) \geq v(\text{LP1}) \geq v(\text{DLP1})$, and that $(x_1^\star \boldsymbol{u}_1 \boldsymbol{u}_1^H, \ldots, x_L^\star \boldsymbol{u}_L \boldsymbol{u}_L^H)$ and $(y_1^\star, \ldots, y_L^\star)$ are optimal for (P2) and (LP1), respectively. ■

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